Optimal Curve Parametrization and an Application to Algebraic ODEs

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Abstract

Rational solutions of algebraic differential equations have been studied a lot, certainly already by Fuchs and Poincaré. We are particularly interested in first-order algebraic differential equations (AODEs) of the form $F(x, y, y') = 0$, where F is a polynomial. The main focus is on rational general solutions, i. e. solutions which are rational functions and depend on an arbitrary transcendental constant. Although in 2010 we have been able to describe a symbolic solution algorithm, which works generically, to this day there is no complete decision algorithm for the existence of rational general solutions.

By modifying the problem slightly, we are able to present an algorithm which decides the existence of a rational general solution in which the arbitrary constant appears rationally. Such a solution we call a strong rational general solution. If there exists one, our algorithm will compute it.

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Outline

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Algebraic Ordinary Differential Equations (AODEs)

An algebraic ordinary differential equation (AODE) is given by

$$F(x, y, y', \dots, y^{(n)}) = 0 \ ,$$

where F is a differential polynomial in $\mathbb{K}[x]\{y\}$ with $\mathbb{K}$ being a differential field and the derivation $'$ being $\frac{d}{dx}$; i.e., $x' = 1$.
We assume that $\mathbb{K}$ is algebraically closed and of characteristic 0.
Such an AODE is autonomous iff the variable of differentiation x does not explicitly appear in F.
We consider first-order AODEs, i.e. the case $n = 1$:

$$F(x, y, y') = 0 \ ,$$

where y' actually appears in F.
W.l.o.g. we may assume that F is irreducible as a polynomial in $\mathbb{K}[x, y, y']$. Then F is also irreducible as an element of $\mathbb{K}(x)[y, y']$.

According to Ritt, the radical differential ideal $\{F\}$ can be decomposed as

$$\{F\} = \underbrace{\left(\{F\} : \frac{\partial F}{\partial y'}\right)}_{general\ \ component} \cap \underbrace{\left\{F, \frac{\partial F}{\partial y'}\right\}}_{singular\ \ component} .$$

S is the separant of F (in general, the derivative of F w.r.t. $y^{(n)}$). Ritt shows that the general component is a prime differential ideal; its generic zero is called a general solution of the AODE $F(x, y, y') = 0$. Such a general solution must contain a transcendental constant c.

J.F. Ritt, *Differential Algebra* (1950)

Problem: Rational general solution of AODE of order 1

given: an AODE $F(x, y, y') = 0$, F irreducible in $\mathbb{K}[x, y, y']$

decide: does this AODE have a rational general solution

find: if so, find it

The study of first-order AODEs dates back to the work of L. Fuchs and H. Poincaré.
Eremenko proved the existence of a degree bound for rational solutions.
Chen and Ma combined an algebro-geometric method with Fuchs' theorem on first-order AODEs without movable critical points.

L.Fuchs, "Über Differentialgleichungen, deren Integrale ..." (1884)
H.Poincaré, "Sur un théorème de M. Fuchs" (1885)
A.Eremenko, "Rational solution of first-order DEs" (1998)
G.Chen, Y.Ma, "Algorithmic reduction and ratinal general solutions of first order algebraic differential equations" (2005)

Example 1 (Example 1.537 in Kamke)

$$F(x, y, y') = (xy' - y)^3 + x^6 y' - 2x^5 y = 0$$

has the rational general solution

$$y(x) = cx(x + c^2) \ ,$$

where c is an arbitrary constant.

E. Kamke, *Differentialgleichungen, Lösungsmethoden und Lösungen* (1997)

The idea is based on an algebro-geometric approach. Similar ideas were already successfully used in Feng/Gao, Ngô/Winkler, Grasegger/Lastra/Sendra/Winkler for computing rational solutions of different classes of differential equations.

Recall that $\mathbb{K}$ is an algebraically closed field.

In contrast to our previous approach, in which we associated a surface in $\mathbb{A}^3(\mathbb{K})$ to a non-autonomous AODE of order 1, here we associate a curve in $\mathbb{A}^2(\mathbb{K}(x))$.
For a given first-order AODE $F(x, y, y') = 0$ over $\mathbb{K}$, we consider the corresponding algebraic curve $\mathcal{C}_F$ defined by the algebraic equation $F(x, y, z) = 0$ over $\mathbb{K}(x)$.

R.Feng, X.-S.Gao, *Proc. ISSAC'04*
L.X.C.Ngô, F.Winkler, *J. Symbolic Computation* (2010)
G.Grasegger, A.Lastra, J.R.Sendra, F.Winkler, *J. Computational and Applied Math.* (2016)

The existence of a strong rational general solution (i.e. the arbitrary constant c appears rationally) implies the existence of a rational parametrization of the algebraic curve $\mathcal{C}_F$.
Algorithms for computing parametrizations of algebraic curves are described, for instance, by Sendra/Winkler/Pérez-Díaz.
Given an optimal parametrization of $\mathcal{C}_F$ over the ground field $\mathbb{K}(x)$, we can transform the AODE to a simpler ODE of the form $\omega' = f(x, \omega)$, which is either linear or a Riccati equation.
This new ODE we call the associated AODE. Then there is a one-to-one correspondence between rational general solutions of the AODE and rational general solutions of its associated AODE.
Using this result it is possible to give a full decision algorithm.

J.R.Sendra, F.Winkler, S.Pérez-Díaz, *Rational Algebraic Curves – A computer algebra approach* (2008)

The main result can be stated in two different ways:

- Given a first-order AODE for which the corresponding curve admits a strong parametrization, we can decide the existence of a rational general solution and compute it in the affirmative case.
- Given any first-order AODE we can decide the existence of a strong rational general solution and compute it in the affirmative case.

Strong rational general solutions

Def. A solution y of the differential equation $F(x, y, y') = 0$ is called a strong rational general solution iff $y = y(x, c) \in \mathbb{K}(x, c) \setminus \mathbb{K}(x)$, where c is a transcendental constant over $\mathbb{K}(x)$.

Example 2 The rational general solution in Example 1 is strong. But the AODE

$$x^3 y'^3 - (3x^2 y - 1) y'^2 + 3xy^2 y' - y^3 + 1 = 0$$

has a rational general solution

$$y(x) = cx + (c^2 + 1)^{\frac{1}{3}} \ ,$$

which is not strong. The curve $\mathcal{C}_F$ has genus 1. So this AODE does not have a strong rational general solution, as we will see.

We give a necessary condition for a first-order AODE to admit a strong rational general solution, i.e. a solution of the form

$$y(x, c) \in \mathbb{K}(x, c) \setminus \mathbb{K}(x) \ , \quad c \text{ a transcendental constant} \ .$$

Theorem 1 *Let* $F \in \mathbb{K}[x, y, z] \setminus \mathbb{K}[x, y]$ *be irreducible.*
If the differential equation $F(x, y, y') = 0$ *has a strong rational solution,*
then its corresponding curve in $\mathbb{A}^2(\overline{\mathbb{K}(x)})$ *is rational (i.e. rationally parametrizable) and admits a parametrization with coefficients in* $\mathbb{K}(x)$*.*

The quality of the parametrization of the curve corresponding to our given AODE will determine the existence of a strong rational general solution.

In general, a rational plane algebraic curve defined over a field K can always be rationally parametrized over a quadratic field extension of K (see Hilbert/Hurwitz). We call a parametrization optimal, if it has coefficients in a field of least extension degree over the field of definition.

By a reasoning similar to the one in Sendra/Winkler/Pérez-Díaz and Hillgarter/Winkler, one can prove the following theorem.

Theorem 2 *Optimal parametrizations of a rational curve over* $\mathbb{K}(x)$ *always have coefficients in* $\mathbb{K}(x)$.

D.Hilbert, A.Hurwitz, "Über die Diophantischen Gleichungen vom Geschlecht Null" (1890)
E.Hillgarter, F.Winkler, "Points on algebraic curves and the parametrization problem" (1998)

Example 1 cont. (Example 1.537 in Kamke) Consider the AODE

$$F(x, y, y') = (xy' - y)^3 + x^6 y' - 2x^5 y = 0 \ .$$

The associated curve $\mathcal{C}_F$ defined by $F(x, y, z) = 0$ can be parametrized as

$$\mathcal{P}(t) = \left(-\frac{t^3x^5 - t^2x^6 + (t-x)^3}{t^3x^5}, \ -\frac{2t^3x^5 - 2t^2x^6 + (t-x)^3}{t^3x^6} \right) \ .$$

This is an optimal parametrization of $\mathcal{C}_F$ over $\overline{\mathbb{Q}}(x)$.

The associated differential equation

Consider a parametrizable first-order AODE $F(x, y, y') = 0$ and assume that an optimal parametrization $\mathcal{P}(t) = (p_1(t), p_2(t)) \in \mathbb{K}(x)(t) \times \mathbb{K}(x)(t)$ of the corresponding curve $\mathcal{C}_F$ is given.
Let $y(x) \in \overline{\mathbb{K}(x)}$ be an algebraic solution of the differential equation.
Then the pair of two algebraic functions $(y(x), y'(x))$ can be seen as a point on the corresponding curve $\mathcal{C}_F$.
Two cases arise.

1. $(y(x), y'(x)) \in \mathcal{C}_F \setminus \mathrm{im}(\mathcal{P})$ (finite set)
2. $(y(x), y'(x)) = \mathcal{P}(\omega(x))$ for some $\omega(x) \in \overline{\mathbb{K}(x)}$.

The algebraic function $\omega(x)$ satisfies the system

$$p_1(x, \omega(x)) = y(x) \ , \quad p_2(x, \omega(x)) = y'(x) \ .$$

Therefore,

$$p_1'(x, \omega(x)) = p_2(x, \omega(x)) \ .$$

By expanding the left hand side, we get

$$\omega'(x) \cdot \frac{\partial p_1}{\partial t}(x, \omega(x)) + \frac{\partial p_1}{\partial x}(x, \omega(x)) \ = \ p_2(x, \omega(x)) \ .$$

Thus, $\omega(x)$ either satisfies the algebraic relation

$$\frac{\partial p_1}{\partial t}(x, \omega(x)) = 0, \quad \frac{\partial p_1}{\partial x}(x, \omega(x)) = p_2(x, \omega(x))$$

or it is an algebraic solution of the quasi-linear differential equation

$$\omega' = \frac{p_2(x, \omega) - \frac{\partial p_1}{\partial x}(x, \omega)}{\frac{\partial p_1}{\partial t}(x, \omega)} \ .$$

This we call the associated differential equation.

Theorem 3 *Let $\mathcal{P}(t)$ be a proper parametrization of the associated curve $\mathcal{C}_F$.*

- *There is a 1-1 correspondence between rational general solutions of the given AODE and rational general solutions of its associated differential equation.*
- *In particular, if $\omega(x)$ is a rational general solution of the associated equation, then $y(x) = p_1(x, \omega(x))$ is a rational general solution of the original AODE.*
- *Conversely, if $y(x)$ is a rational general solution of the original AODE, then $\omega(x) = \mathcal{P}^{-1}(y(x), y'(x))$ is a rational general solution of the associated equation.*

As we have seen above, for finding rational solutions of a parametrizable first-order AODE it suffices to work with the class of quasi-linear first-order ODEs.

If we look for rational general solutions, the situation is even much more restricted.

In fact, Behloul and Cheng proved that if a quasi-linear differential equation has infinitely many rational solutions, then it must be either a linear differential equation or a Riccati equation.

So, from Theorem 3 and the result of Behloul and Cheng we get the following.

D. Behloul, S.S. Cheng, "Computation of rational solutions for a first-order nonlinear differential equations" (2011)

Theorem 4 *Let $F(x, y, y') = 0$ be a first-order AODE.*

1. *If $F(x, y, y') = 0$ has a strong rational general solution, then $\mathcal{C}_F$ is parametrizable and the associated differential equation is a Riccati equation of the form*

$$\omega' = a_0(x) + a_1(x)\omega + a_2(x)\omega^2 \ ,$$

for some $a_0, a_1, a_2 \in \mathbb{K}(x)$.

2. *If $F(x, y, y') = 0$ has a rational general solution and $\mathcal{C}_F$ is parametrizable, then the associated differential equation is of the form above.*

In the proof of this theorem, we use a result by Fuchs (or Behloul and Cheng), which says: "If a quasi-linear ODE $y' = f(x, y)$, where f is a rational function in x and y, has a rational general solution, then it must be a linear or Riccati equation". In case the parametrization is not optimal, it may contain a square root of x (or an algebraic function in x which is not rational). This square root then also appears in the associated quasi linear ODE. In this case, we cannot apply Fuchs' result.

F. Schwarz shows that if a Riccati equation has 3 special rational solutions, then it has a strong rational general solution. Combining this with Theorem 4 we get

Corollary *If a parametrizable first-order AODE has a rational general solution, then it has a strong rational general solution.*

F.Schwarz, *Algorithmic Lie theory for solving ordinary differential equations* (2008);
Cor. 2.1, p.18

We are looking for rational general solutions of first-order AODEs. The remaining problem is to determine a rational general solution of the associated equation in Theorem 4.

In the case $a_2 = 0$, it is a linear differential equation of degree 1 which can be easily solved by integration.

In the case $a_2 \neq 0$, it is a classical Riccati equation.

Kovacic presents a full algorithm for determining all rational solutions of a Riccati equation. Note that for a Riccati equation, the notions of rational general solutions and strong rational general solutions coincide.

Chen and Ma modify Kovacic's algorithm slightly to determine only strong rational general solutions.

The key for turning this approach into a decision algorithm is Theorem 2: a rational curve over $\mathbb{K}(x)$ can be parametrized with coefficients in $\mathbb{K}(x)$.

M.M.Kovacic, An algorithm for solving second order linear homogeneous differential equations, J.Symb.Computation 2/2, 3–43, Section 3.1

The algorithm

We present a full algorithm which computes for a given first-order AODE a strong rational general solution, if it exists. Otherwise it decides that such a solution cannot exist.

Algorithm STRONG-RAT-GEN-SOLVE

Input: a first-order AODE, $F(x, y, y') = 0$;
$F \in \mathbb{K}[x, y, z] \setminus \mathbb{K}[x, y]$ irreducible

Determine: a strong rational general solution $y(x)$,
or "No strong rational general solution exists"

1. **if** genus of $\mathcal{C}_F$ is not zero **then goto** (5)
2. Compute optimal parametrization of $\mathcal{C}_F$, say $(p_1(x, t), p_2(x, t))$, with coefficients in $\mathbb{K}(x, t)$
3. Determine associated differential equation

$$f(x, t) := \frac{p_2(x, t) - \frac{\partial}{\partial x} p_1(x, t)}{\frac{\partial}{\partial t} p_1(x, t)}$$

4. **if** $f(x, t)$ has the form $a_0(x) + a_1(x)t + a_2(x)t^2$
   for some $a_0, a_1, a_2 \in \mathbb{K}(x)$
   and the linear or Riccati equation $\omega' = f(x, \omega)$;
   has a rational general solution $\omega = \omega(x)$
   then return $y(x) = p_1(x, \omega(x))$
5. **return** "No strong rational general solution exists".

Example 1 cont. (Example 1.537 in Kamke) Consider the AODE

$$F(x,y,y') = (xy'-y)^3 + x^6y' - 2x^5y = 0 \ .$$

The associated curve $\mathcal{C}_F$ defined by $F(x,y,z)=0$ can be parametrized as

$$\mathcal{P}(t) = \left(-\frac{t^3x^5 - t^2x^6 + (t-x)^3}{t^3x^5},\ -\frac{2t^3x^5 - 2t^2x^6 + (t-x)^3}{t^3x^6} \right) \ .$$

Therefore, the associated differential equation w.r.t. $\mathcal{P}$ is

$$\omega' = \frac{1}{x^2}\omega(2\omega - x) \ ,$$

which is a Riccati equation. Kovacic's algorithm gives us the rational general solution $\omega(x) = \frac{x}{1+cx^2}$.
Hence, the original AODE $F(x,y,y')=0$ has the strong rational general solution

$$y(x) = cx(x+c^2) \ .$$

Observe, that this is just an arbitrary example from the collection of Kamke. In total around 64 percent of the listed ODEs there are AODEs and almost all of them are parametrizable and hence suitable for algorithm STRONG-RAT-GEN-SOLVE. The remaining ODEs without strong parametrization are either reducible or the corresponding curve has higher genus. For more details see our technical report RISC 15-19.

Conclusion

We have presented an algorithm for deciding whether a strong rational general solution of a first-order AODE exists. In the affirmative case the algorithm also computes such a solution.

- Given any first-order AODE we can decide the existence of a strong rational general solution and compute it in the affirmative case.

The algorithm is based on optimal curve parametrizations over the field of rational functions.

Thank you for your attention!