Closed Formula for Univariate Subresultants in Multiple Roots

Marcelo Valdettaro

Coauthors: Carlos D'Andrea, Teresa Krick and Agnes Szanto

Universidad de Buenos Aires

Subresultant (Jacobi 1836 - Sylvester 1839)

- $f(x) = \sum_{i=0}^{m} f_i x^i, \; g(x) = \sum_{i=0}^{n} g_i x^i$
- $0 \le d \le \min\{m, n\}$ if $m \ne n$ or $0 \le d < m = n$

$$\operatorname{Sres}_d(f,g)(x) = \det \overbrace{\left|\begin{array}{ccccc} f_m & \cdots & \cdots & f_{d+1-(n-d-1)} & x^{n-d-1}f(x) \\ & \ddots & & \vdots & \vdots \\ & & f_m \;\; \cdots & f_{d+1} & f(x) \\ \hline g_n & \cdots & \cdots & g_{d+1-(m-d-1)} & x^{m-d-1}g(x) \\ & \ddots & & \vdots & \vdots \\ & & g_n \;\; \cdots & g_{d+1} & g(x) \end{array}\right|}^{m+n-2d} \begin{array}{l} n-d \\ \\ m-d \end{array}$$

Sylvester sums (1840)

- $|A| = m$, $|B| = n$, $0 \leq d \leq m$

$$\mathrm{Syl}_{d,0}(A,B)(x) = \sum_{\substack{A_1 \cup A_2 = A \\ |A_1| = d,\, |A_2| = m-d}} \frac{\mathcal{R}(A_2,B)\mathcal{R}(x,A_1)}{\mathcal{R}(A_1,A_2)}$$

Where $\mathcal{R}(Y,Z) = \prod_{\substack{y \in Y \\ z \in Z}} (y - z)$, $\mathcal{R}(X,Y) = 1$ if $X = \emptyset$ or $Y = \emptyset$

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Where $\mathcal{R}(Y,Z) = \prod_{\substack{y \in Y \\ z \in Z}} (y - z)$, $\mathcal{R}(X,Y) = 1$ if $X = \emptyset$ or $Y = \emptyset$

- Also, for $0 \le d \le n$

$$\mathrm{Syl}_{0,d}(A,B)(x) = \sum_{\substack{B_1 \cup B_2 = B \\ |B_1| = d,\, |B_2| = n-d}} \frac{\mathcal{R}(A, B_2)\mathcal{R}(x, B_1)}{\mathcal{R}(B_1, B_2)}$$

Sylvester 1853

- $f = \prod_{\alpha \in A} (x - \alpha) = \mathcal{R}(x, A)$
- $g = \prod_{\beta \in B} (x - \beta) = \mathcal{R}(x, B)$

$$\mathrm{Sres}_d(f, g)(x) = \pm \mathrm{Syl}_{d,0}(A, B)(x) = \pm \mathrm{Syl}_{0,d}(A, B)(x)$$

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This is

$$\sum_{\substack{A_1 \cup A_2 = A \\ |A_1| = d,\ |A_2| = m-d}} \frac{\mathcal{R}(A_2, B)\mathcal{R}(x, A_1)}{\mathcal{R}(A_1, A_2)} = \pm \sum_{\substack{B_1 \cup B_2 = B \\ |B_1| = d,\ |B_2| = n-d}} \frac{\mathcal{R}(A, B_2)\mathcal{R}(x, B_1)}{\mathcal{R}(B_1, B_2)}$$

Theorem

For enough roots (or big d):

- $f = \mathcal{R}(x, A)$, $g = \mathcal{R}(x, B)$, with A, B multisets
- $\overline{A}$, $\overline{B}$, sets of roots of f and g, $|\overline{A}| = \overline{m}$, $|\overline{B}| = \overline{n}$
- $\overline{m} + \overline{n} \geq m + n - d$ (i.e. $d \geq m - \overline{m} + n - \overline{n}$)

$$\operatorname{Sres}_d(f, g)(x) = \pm \sum_{\substack{A_1 \cup A_2 = \overline{A},\, B_1 \cup B_2 = \overline{B} \\ |A_1| = d - (m - \overline{m}),\, |A_2| = m - d \\ |B_1| = m - \overline{m},\, |B_2| = \overline{n} - (m - \overline{m})}} \frac{\mathcal{R}(A \backslash \overline{A}, B_2)\mathcal{R}(A_2, B \backslash B_1)\mathcal{R}(x, A_1)\mathcal{R}(x, B_1)}{\mathcal{R}(A_2, A_1)\mathcal{R}(B_2, B_1)}$$

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For the general case: A more complicated formula involving Schur polynomials

Example 1

- $f = (x - \alpha_1)(x - \alpha_2)^2$, $g = (x - \beta_1)^2$
- $A = (\alpha_1, \alpha_2, \alpha_2)$, $\overline{A} = \{\alpha_1, \alpha_2\}$
- $B = (\beta_1, \beta_1)$, $\overline{B} = \{\beta_1\}$
- $d = 2 = m - \overline{m} + n - \overline{n}$

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$$
\begin{aligned}
&\frac{(\alpha_2 - \beta_1)(x - \alpha_1)(x - \beta_1)}{\alpha_2 - \alpha_1} + \frac{(\alpha_1 - \beta_1)(x - \alpha_2)(x - \beta_1)}{\alpha_1 - \alpha_2} \\
&= \frac{\big((\alpha_2 - \beta_1)(x - \alpha_1) - (\alpha_1 - \beta_1)(x - \alpha_2)\big)(x - \beta_1)}{\alpha_2 - \alpha_1} \\
&= (x - \beta_1)(x - \beta_1) \\
&= g(x) \\
&= \mathrm{Sres}_2(f, g)(x)
\end{aligned}
$$

Example 2

- $f = (x - \alpha_1)(x - \alpha_2)^2$, $g = (x - \beta_1)^3$
- $A = (\alpha_1, \alpha_2, \alpha_2)$, $\overline{A} = \{\alpha_1, \alpha_2\}$
- $B = (\beta_1, \beta_1, \beta_1)$, $\overline{B} = \{\beta_1\}$
- $d = 2 < 3 = m - \overline{m} + n - \overline{n}$

$$\frac{(\alpha_2 - \beta_1)(\alpha_2 - \beta_1)(x - \alpha_1)(x - \beta_1)}{\alpha_1 - \alpha_2} + \frac{(\alpha_1 - \beta_1)(\alpha_1 - \beta_1)(x - \alpha_2)(x - \beta_1)}{\alpha_2 - \alpha_1}$$

$$\neq \mathrm{Sres}_2(f, g)(x)$$

Example 2

- $f = (x - \alpha_1)(x - \alpha_2)^2$, $g = (x - \beta_1)^3$
- $A = (\alpha_1, \alpha_2, \alpha_2)$, $\overline{A} = \{\alpha_1, \alpha_2\}$
- $B = (\beta_1, \beta_1, \beta_1)$, $\overline{B} = \{\beta_1\}$
- $d = 2 < 3 = m - \overline{m} + n - \overline{n}$

$$\frac{(\alpha_2 - \beta_1)(\alpha_2 - \beta_1)(x - \alpha_1)(x - \beta_1)}{\alpha_1 - \alpha_2} + \frac{(\alpha_1 - \beta_1)(\alpha_1 - \beta_1)(x - \alpha_2)(x - \beta_1)}{\alpha_2 - \alpha_1}$$

$$-(\alpha_2 - \beta_1)(x - \alpha_1)(x - \alpha_2) = \mathrm{Sres}_2(f, g)(x)$$

Exchange lemma

- $|A| = m$, $|B| = n$, $0 \leq d \leq \min\{m, n\}$, $|X| \leq m - d + n - d$

$$\sum_{\substack{A_1 \cup A_2 = A \\ |A_1| = d,\, |A_2| = m-d}} \frac{\mathcal{R}(A_2, B)\mathcal{R}(X, A_1)}{\mathcal{R}(A_2, A_1)} = \pm \sum_{\substack{B_1 \cup B_2 = B \\ |B_1| = d,\, |B_2| = n-d}} \frac{\mathcal{R}(A, B_2)\mathcal{R}(X, B_1)}{\mathcal{R}(B_2, B_1)}$$

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- $|A| = m$, $|B| = n$, $0 \le d \le \min\{m, n\}$, $|X| \le m - d + n - d$

$$\sum_{\substack{A_1 \cup A_2 = A \\ |A_1| = d,\ |A_2| = m-d}} \frac{\mathcal{R}(A_2, B)\mathcal{R}(X, A_1)}{\mathcal{R}(A_2, A_1)} = \pm \sum_{\substack{B_1 \cup B_2 = B \\ |B_1| = d,\ |B_2| = n-d}} \frac{\mathcal{R}(A, B_2)\mathcal{R}(X, B_1)}{\mathcal{R}(B_2, B_1)}$$

- When $d = n$, if $m \ge |X| + d$

$$\sum_{\substack{A_1 \cup A_2 = A \\ |A_1| = d,\ |A_2| = m-d}} \frac{\mathcal{R}(A_2, B)\mathcal{R}(X, A_1)}{\mathcal{R}(A_2, A_1)} = \pm \mathcal{R}(X, B)$$

(The case $m = |X| + d$ is an equality due to A. Lascoux)

Exchange lemma

For three sets, if $|E| \geq \max\{m+n-d, |X|+d\}$

$$\sum_{\substack{A_1 \cup A_2 = A \\ |A_1|=d,\, |A_2|=m-d}} \frac{\mathcal{R}(A_2, B)\mathcal{R}(X, A_1)}{\mathcal{R}(A_2, A_1)} = \pm \sum_{\substack{E_1 \cup E_2 \cup E_3 = E \\ |E_1|=d,\, |E_2|=m-d \\ |E_3|=|E|-m}} \frac{\mathcal{R}(E_3, A)\mathcal{R}(E_2, B)\mathcal{R}(X, E_1)}{\mathcal{R}(E_3, E_2)\mathcal{R}(E_3, E_1)\mathcal{R}(E_2, E_1)}$$

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This generalizes a formula due to F. Apéry and J.-P. Jouanolou (2006):

$$\mathrm{Sres}_d(f,g)(x) = \pm \sum_{\substack{E_1 \cup E_2 \cup E_3 = E \\ |E_1|=d,\, |E_2|=m-d \\ |E_3|=n-d}} \frac{\mathcal{R}(E_3,A)\mathcal{R}(E_2,B)\mathcal{R}(x,E_1)}{\mathcal{R}(E_3,E_2)\mathcal{R}(E_3,E_1)\mathcal{R}(E_2,E_1)}$$

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If there are enough roots (d is big enough) take $E = \overline{A} \cup \overline{B}$

What if there are not enough roots (d is small)?

$$\sum_{\substack{A_1 \cup A_2 = A \\ |A_1|=d,\, |A_2|=m-d}} \frac{\mathcal{R}(A_2,B)\mathcal{R}(x,A_1)}{\mathcal{R}(A_2,A_1)} = \pm \sum_{\substack{E_1 \cup E_2 \cup E_3 = E \\ |E_1|=d,\, |E_2|=m-d \\ |E_3|=n-d}} \frac{\mathcal{R}(E_3,A)\mathcal{R}(E_2,B)\mathcal{R}(x,E_1)}{\mathcal{R}(E_3,E_2)\mathcal{R}(E_3,E_1)\mathcal{R}(E_2,E_1)}$$

- Set $E = \overline{A} \cup \overline{B} \cup T$ with $|T| = m + n - d - \overline{m} - \overline{n}$
- Multiply both sides by $V(T)\mathcal{R}(T, \overline{A} \cup \overline{B})$
- Take a suitable coefficient in the variables $t_i \in T$

Thank you!

The formula for the small d case

- $f = \mathcal{R}(x, A)$, $g = \mathcal{R}(x, B)$, with A, B multisets
- $\overline{A}$, $\overline{B}$, sets of roots of f and g, $|\overline{A}| = \overline{m}$, $|\overline{B}| = \overline{n}$
- $\overline{m} + \overline{n} < m + n - d$ (i.e. $d < m - \overline{m} + n - \overline{n}$)
- $r = m + n - d - \overline{m} - \overline{n}$

$$\mathrm{Sres}_d(f,g)(x) = \sum_{\substack{R_1 \sqcup R_2 \sqcup R_3 = \{1,\dots,r\} \\ |R_i| =: r_i}} (-1)^{\sigma_R} \sum_{\substack{A_1 \cup A_2 = \overline{A},\ B_1 \cup B_2 = \overline{B} \\ |A_1| = d-(m-\overline{m})+r_3,\ |A_2| = m-d-r_3 \\ |B_1| = d-(n-\overline{n})+r_2,\ |B_2| = n-d-r_2}} \frac{\mathcal{R}(A\backslash\overline{A}, B_2)\mathcal{R}(A_2, B\backslash B_1)\mathcal{R}(x, A_1)\mathcal{R}(x, B_1)}{\mathcal{R}(A_2, A_1)\mathcal{R}(B_2, B_1)} \cdot S^{(R_3)}_{m+n-d}(A \cup B_2) S^{(R_2)}_{m+n-d}(B \cup A_2) S^{(R_1)}_{d+1}(A_1 \cup B_1 \cup x)$$