The bifurcation locus of polynomial maps

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Bifurcation locus

The bifurcation locus of a polynomial mapping $F : \mathbb{R}^n \to \mathbb{R}^p$, $n \geq p$, is the minimal set of points $B(F) \subset \mathbb{R}^p$ outside which the mapping is a C^∞ locally trivial fibration. Unlike the local setting, the critical locus $\mathrm{Sing} F$ is not the only obstruction to the existence of fibrations in the global setting.

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There is an example by Pinchuk of a polynomial mapping $F : \mathbb{R}^2 \to \mathbb{R}^2$ where $\mathrm{Sing} F = \emptyset$ but $B(F) \neq \emptyset$, thus providing a counter-example to the strong Jacobian Conjecture over the reals.

The Jacobian problem remains nevertheless open over $\mathbb{C}$, even in two variables.

Real versus complex setting

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A complex polynomial function $f : \mathbb{C}^2 \to \mathbb{C}$ which is a trivial fibration is actually equivalent to a linear function, modulo automorphisms of $\mathbb{C}^2$.

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This is again not true over $\mathbb{R}$:

Example

The polynomial function $g : \mathbb{R}^2 \to \mathbb{R}$, $g(x, y) = y(x^2 + 1)$ is a component of a diffeomorphism, fact that one can see by using the change of variables $(x, y) \mapsto (x, \frac{y}{x^2+1})$. Therefore g is a trivial fibration.
However, g cannot be linearised by a *polynomial* automorphism.

The Euler characteristic test

The following result was found in the 1970's by Suzuki:

Let $f : \mathbb{C}^2 \to \mathbb{C}$ *be a polynomial function and let* $a \in \mathbb{C} \setminus f(\mathrm{Sing} f)$*.*
Then $a \notin B(f)$ *if and only if the Euler characteristic of the fibres* $\chi(f^{-1}(t))$ *is constant for* t *varying in some neighbourhood of* a*.*

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Its real counterpart came out much later. It appears that for polynomial functions $\mathbb{R}^2 \to \mathbb{R}$ the constancy of the Euler characteristic of the fibres is not sufficient.
The phenomena that occur at infinity, i.e. the "splitting" and the "vanishing" of components of fibres, may not be detected by the Euler characteristic.

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Theorem (T-Zaharia 1999)

Let X be a real algebraic nonsingular surface and let $\tau : X \to \mathbb{R}$ be an algebraic map. Let $a \in \mathrm{Im}\tau$ be a regular value of τ, and let $X_t := F^{-1}(t)$. Then $a \notin B(\tau)$ if and only if:

1. *the Euler characteristic $\chi(X_t)$ is constant when t varies in some neighbourhood of a, and*
2. *there is no component of X_t which vanishes at infinity as t tends to a.*

One moreover shows that the above criterion (a)+(b) is equivalent to the following:

(c) *the Betti numbers of* X_t *are constant for* t *in some neighborhood of* a, and

(d) *there is no component of* X_t *which splits at infinity as* t *tends to* a.

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The following theorem represents the extension of the above result to algebraic families of curves of more than one parameter.

Theorem (Joița-T 2017)

Let $X \subset \mathbb{R}^m$ *be a real nonsingular irreducible algebraic set of dimension* $n \geq 3$ *and let* $F : X \to \mathbb{R}^{n-1}$ *be an algebraic map. Let* a *be an interior point of the set* $\mathrm{Im}F \setminus \overline{F(\mathrm{Sing}F)} \subset \mathbb{R}^{n-1}$ *and let* $X_t := F^{-1}(t)$*. Then* $a \notin B(F)$ *if and only if the following two conditions are satisfied:*

1. *the Euler characteristic* $\chi(X_t)$ *is constant when* t *varies within some neighbourhood of* a*, and*
2. *there is no component* X_t^j *of* X_t *which vanishes at infinity as* t *tends to* a*.*

The above criterion (1)+(2) may be replaced by (a)+(b), where:

(a) *the Betti numbers of* X_b *are constant for* b *in some neighbourhood of* a*, and*

(b) *there is no splitting at infinity at* a*.*

The polynomial

$$f(x,y) := x^2y^3(y^2-25)^2 + 2xy(y^2-25)(y+25) - (y^4+y^3-50y^2-51y+575)$$

has the property that 0 is a regular value which is atypical, but the Betti numbers of the fibres $f^{-1}(t)$ are constant, for $|t|$ small enough. Namely, all these fibers have 5 non–compact connected components.
Two are vanishing and two are splitting as $t \to 0$.

Detecting bifurcation values by the Milnor set

$F : \mathbb{R}^n \to \mathbb{R}^p$ polynomial map.
In more than two variables, over the last 20 years one could only estimate $B(F)$ by supersets $A \supset B(F)$ according to certain *regularity conditions at infinity*. The bifurcation set $B(F)$ was shown to be precisely detectable only if $p = 1$ and $n = 2$. The similar situation holds over the complex field.

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In case $n \geq 3$, $n > p$, the bifurcation non-critical locus $B(F) \setminus F(\mathrm{Sing} F)$ is included in the set of "ρ-nonregular values at infinity". The *ρ-regularity* is a Milnor type condition that controls the transversality of the fibres of F to the spheres centred at $c \in \mathbb{R}^n$, more precisely:

Definition

Let $\rho_c : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ denote the Euclidian distance function to the point $c \in \mathbb{R}^n$. We call *Milnor set of* (F, ρ_c) the critical set of the mapping $(F, \rho_c) : \mathbb{R}^n \to \mathbb{R}^{p+1}$ and denote it by $M_c(F)$.
We call:

$$S_c(F) := \{t_0 \in \mathbb{R}^p \mid \exists \{x_j\}_{j \in \mathbb{N}} \subset M_c(F), \lim_{j\to\infty} \|x_j\| = \infty \text{ and } \lim_{j\to\infty} F(x_j) = t_0\}$$

the set of ρ_c*-nonregular values at infinity*. If $t_0 \notin S_c(F)$ we say that t_0 is ρ_c*-regular at infinity*. We set $S_\infty(F) := \bigcap_{c\in\mathbb{R}^n} S_c(F)$.

In case of polynomials $f : \mathbb{C}^2 \to \mathbb{C}$ the following characterisation has been proved in the 1980's:

Let $a \in \mathbb{C} \setminus f(\mathrm{Sing} f)$. Then $a \in B(f)$ if and only if $a \in S_0(f)$.

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This is not true anymore over the reals, as shown by the following example by Tibar-Zaharia 1999:

$f : \mathbb{R}^2 \to \mathbb{R}$, $f(x, y) = y(2x^2y^2 - 9xy + 12)$,
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There is some open dense set $\Omega_f \subset \mathbb{R}^2$ such that for $c \in \Omega_f$ the Milnor set $M_c(f)$ is a curve (or it is empty). For such a point $c \in \Omega_f$ one counts the number $\#[X_t^j \cap M_c(f)]$ of points of intersection of the connected components X_t^j of the fibres X_t with the curve $M_c(f)$. The following criterion holds:

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Let $a \in \mathbb{R} \setminus f(\mathrm{Sing} f)$. Then $a \in B(f)$ if and only if $a \in S_c(f)$ and $\lim_{t \to a} \#[X_t^j \cap M_c(f)] \not\equiv 0 \pmod 2$ for some family of connected components X_t^j of X_t.

Multiparameter families of complex curves

Suzuki's Euler test for $f : \mathbb{C}^2 \to \mathbb{C}$ is not anymore sufficient if we replace $\mathbb{C}^2$ by an affine surface X (examples by Zaidenberg and Gurjar, Miyanishi), or in case of polynomial maps with $n > p \geq 2$. This led to the natural question (Gurjar 2012):

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Theorem (Joița-T)

Let $F : \mathbb{C}^{n+1} \to \mathbb{C}^n$ be a polynomial map. Let λ be a point in the interior of the set $\mathrm{Im} F \setminus \overline{F(\mathrm{Sing} F)} \subset \mathbb{C}^n$. Then $\lambda \notin B(F)$ if and only if the Euler characteristic of the fibres $F^{-1}(t)$ is constant for t varying in some neighborhood of λ and no connected component of $F^{-1}(t)$ is vanishing at infinity as $t \to \lambda$.

To define “vanishing components”, we consider the more general situation of a holomorphic map $p : M \to B$ between connected complex manifolds with $\dim_{\mathbb{C}} M = \dim_{\mathbb{C}} B + 1$, and let $\lambda \in \mathrm{Im} p$ be a regular value of p. The fiber $p^{-1}(t)$ is a complex manifold of dimension 1 and may be not connected; we assume that it has finitely many connected components. We then denote by C_t some connected component of the fiber $p^{-1}(t)$.

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Definition

We say that there are *vanishing components at infinity when t tends to* λ if there is a sequence of points $t_k \in B$, $t_k \to \lambda$ such that for some choice of a connected component C_{t_k} of $p^{-1}(t_k)$ the sequence of sets $\{C_{t_k}\}_{k\in\mathbb{N}}$ is *locally finite*, i.e., for any compact $K \subset M$, there is an integer $p_K \in \mathbb{N}$ such that $\forall q \geq p_K$, $C_{t_q} \cap K = \emptyset$. If *no connected component of* $p^{-1}(t)$ *is vanishing at infinity when* $t \to \lambda$ then we say that *one has the property (NV) at* λ.

A more effective way to define (NV) is as follows: we denote by M_b the fibre $p^{-1}(b)$. Let $M_b = \sqcup_j M_b^j$ be the decomposition of the fibre M_b into connected components. Let also $\varphi : M \to \mathbb{R}$ be a continuous exhaustion function, i.e. $\{x \in M : \varphi(x) \leq r\}$ is compact for every $r \in \mathbb{R}$ (for example, if $M = \mathbb{C}^n$, one we may take $\varphi(x) = \|x\|$). We define:

$$\mu(b) := \max_j \inf_{x \in M_b^j} \varphi(x)$$

Then "vanishing component at infinity when $t \to \lambda$" means that there exists a sequence of points $t_k \in B$, $t_k \to \lambda$, such that $\lim_{k\to\infty} \mu(t_k) = \infty$.

The phenomenon of “vanishing of components” has been studied in the real setting of polynomials $\mathbb{R}^2 \to \mathbb{R}$, as we have seen before. It turns out that this is related to the “vanishing cycles” and “emerging cycles” introduced for independent reasons by Meigniez around 1985. In another stream, “vanishing cycles at infinity” have been studied in the context of complex polynomial functions by Parusinski, Neumann, Dimca, Siersma-Tibar, Nemethi, Sabbah etc.

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It was first shown in the real setting that these two phenomena may happen simultaneously while *the Betti numbers are locally constant*. We present here an example in the complex setting where all these phenomena occur. It proves at the same time that the non-vanishing hypothesis of our Theorem is necessary.

[Tibar-Joita] Let $F : \mathbb{C}^3 \to \mathbb{C}^2$ be defined by

$$F(x, y, z) = \left(x, [(x-1)(xz + y^2) + 1][x(xz + y^2) - 1]\right).$$

The singular locus of F is the union of the z-axis, a curve and a surface, the last two having the same image by F. One then checks that $(0, 0)$ is an interior point of $\mathbb{C}^2 \setminus \mathrm{Sing} f$.

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If $(a,b) \in \mathbb{C}^2$ is close enough to $(0,0)$ and $a \neq 0$ then

$f^{-1}(a,b) = \{(x,y,z) \in \mathbb{C}^3 \mid x = a,\ az+y^2 = c_1\} \cup \{(x,y,z) \in \mathbb{C}^3 \mid x = a,\ az+y^2 = c_2\}$

where c_1 and c_2 are the two distinct roots of the equation $[(a-1)\lambda+1][a\lambda-1] = a(a-1)\lambda^2+\lambda-1 = b$.

Therefore $f^{-1}(a,b) \simeq \mathbb{C} \sqcup \mathbb{C}$.

If b is close enough to zero then

$$f^{-1}(0,b) = \{(x,y,z) \in \mathbb{C}^3 \mid x=0,\ y=d_1\} \cup \{(x,y,z) \in \mathbb{C}^3 \mid x=0,\ y=d_2\}$$

where d_1 and d_2 are the two roots of $y^2 = b^2+1$; thus $f^{-1}(a,b) \simeq \mathbb{C} \sqcup \mathbb{C}$.

All fibers of f in a small neighborhood of $(0,0) \in \mathbb{C}^2$ have therefore exactly two connected components isomorphic to $\mathbb{C}$. On the other hand, when $a \to 0$ one component vanishes at infinity and the other one splits into two copies of $\mathbb{C}$, which shows that f is not locally trivial at $(0,0)$ whereas its fibres are abstractly diffeomorphic and even isomorphic.

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In order to prove our Theorem we show the following more general result:

Theorem

Let $p : M \to B$ be a holomorphic map between connected complex manifolds, where M is Stein and $\dim M = \dim B + 1$. We asume that the Betti numbers $b_0(t)$ and $b_1(t)$ of all the fibers $p^{-1}(t)$ are finite. Let λ be a point in the interior of the set $\mathrm{Imp} \setminus \overline{p(\mathrm{Sing}\, p)} \subset B$. Then $\lambda \notin B(p)$ if and only if the Euler characteristic of the fibres is constant for t varying in some neighborhood of λ and no connected component of $p^{-1}(t)$ is vanishing at infinity when $t \to \lambda$.

This is in turn based on the following result for connected fibres, the proof of which relies on deep results by Ilyashenko and Meigniez.

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Theorem

Let $p : M \to B$ be a holomorphic map between connected complex manifolds which is a submersion, where M is Stein with $\dim M = \dim B + 1$. Assume that all the fibres of p are connected and have the same finite first Betti number $b_1 < \infty$. Then p is a trivial fibration.

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The condition of non-vanishing components at infinity is then employed in order to reduce the general case to the above Theorem.

Effectivity aspects: bifurcation locus for complex polynomial functions

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$n \geq 2$: $B(f)$ might be larger than the set $f(\mathrm{Sing} f)$ of critical values of f; it also contains the set $B_\infty(f)$ of *bifurcation points at infinity*, namely $B(f) = f(\mathrm{Sing} f) \cup B_\infty(f)$.

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Roughly speaking, $B_\infty(f)$ consists of points at which the restriction of f to a neighborhood of infinity (i.e. outside a large enough ball) is not a locally trivial bundle.

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Theorem ($n = 2$)

$a \in B_\infty(f)$ if and only if there exists a sequence of points $(p_k)_{k\in\mathbb{N}} \subset \mathbb{C}^2$ such that $\|p_k\| \to \infty$, $f(p_k) \to a$ and $\mathrm{grad} f(p_k) \to 0$ as $k \to \infty$.

Bounds

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Theorem (Gwoździewicz - Płoski, 2001)

If $\dim \mathrm{Sing} f \leq 0$ *then* $\#B_\infty(f) \leq \max\{1, d-3\}$.

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Estimation in terms of the degree $d = \deg f$:

Theorem (Gwoździewicz - Płoski, 2001)

If $\dim \mathrm{Sing} f \leq 0$ *then* $\#B_\infty(f) \leq \max\{1, d-3\}$.

For any singular locus $\mathrm{Sing} f$:

$$\#B_\infty(f) \leq d - 1.$$

More recently:

Theorem (Gwoździewicz, 2013)

$\#\{B_\infty(f) \setminus \{0\}\} \leq$ *number of branches at infinity of the red. fibre* $f^{-1}(0)$.

$\nu_a :=$ the number of branches at infinity of the (reduced) fiber $f^{-1}(a)$.
This number is equal to ν_{gen} for all values $a \in \mathbb{C}$ except finitely many for which one may have either $\nu_a < \nu_{\text{gen}}$ or $\nu_a > \nu_{\text{gen}}$.

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Theorem (Jelonek-T, 2015)

Let $f : \mathbb{C}^2 \to \mathbb{C}$ *be a polynomial function of degree* d*. Then:*

1. $\#B_\infty(f) \leq \min\{\nu_{\text{gen}}, \nu_{\min} + 1\}$.

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In case $\nu_{\text{gen}} > \frac{d}{2}$, we moreover have:

(d) $\#B_\infty(f) \leq \min\{\nu_{\text{gen}} - 1, \nu_{\text{min}}\}$.

(e) $\#\{a \in \mathbb{C} \mid \nu_a > \nu_{\text{gen}}\} \leq \nu_{\text{min}} - 1$.

Theorem (Gurjar, Miyanishi)

Let X be a normal affine surface with a $\mathbb{C}$-fibration $f : X \to B$, where B is a smooth curve. Then:

1. *X has at most cyclic quotient singularities.*
2. *every fiber of f is a disjoint union of curves isomorphic to $\mathbb{C}$.*
3. *a component of a fiber of f contains at most one singular point of X. If a component of a fiber occurs with multiplicity 1 in the scheme-theoretic fiber, then no singular point of X lies on this component.*

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Corollary

Let X be a normal affine surface, which contains a cylinder-like open subset U, i.e. there exists a smooth curve C such that $U \cong \mathbb{C} \times C$.

Then the set $X \setminus U$ is a disjoint union of curves isomorphic to $\mathbb{C}$. Moreover, every connected component l_i of this set contains at most one singular point of X.

Bifurcation locus in higher dimensions

To control the bifurcation set $B_\infty(f)$ of a polynomial $f : \mathbb{C}^n \to \mathbb{C}$ of degree $d \geq 2$ one can use the set of *asymptotic critical values* (or *Malgrange non-regular values*):

$$K_\infty(f) := \{y \in \mathbb{C} \mid \exists\ (x_l)_{l\in\mathbb{N}}, \|x_l\| \to \infty, \quad f(x_l) \to y \text{ and } \|x_l\| \|\mathrm{grad} f(x_l)\| \to 0\}.$$

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$TK_\infty(f) :=$ trivial Malgrange non-regular values which come from the critical points of f, i.e. $\exists\ x_l \to \infty$ such that $x_l \in \mathrm{Sing}(f)$ and $f(x_l) \to y$

$NK_\infty(f) := K_\infty(f) \setminus TK_\infty(f)$ the set of non-trival Malgrange non-regular values.

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$$TK_\infty(f) \subset f(\mathrm{Sing} f)$$

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Iterated polars

We characterize the set $NK_\infty(f)$ by using a series of *polar curves* and their relation to Malgrange non-regularity via the *t*-regularity.

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Let $\{x_1, \ldots, x_n\}$ be a *generic system of coordinates* of $\mathbb{C}^n$.
Let us consider the successive restrictions of f to the affine hyperplanes:

$$f_0 := f, \quad f_1 := f_{|x_1=0}, \ \ldots, \ f_{n-2} := f_{|x_1=\cdots=x_{n-2}=0},$$

and the corresponding generic polar curves $\Gamma(x_i, f_{i-1})$, for $i = 1, \ldots, n-1$.

$g : X \to Y$ dominant, generically finite polynomial map of smooth affine varieties.

$Y \supset J_g :=$ *the non-properness set* of the mapping g (a.k.a. *Jelonek set*).
g is not proper at y if there is a sequence $x_l \to \infty$ such that $g(x_l) \to y$.

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Theorem (Jelonek, 1993)

Let $X \subset \mathbb{C}^k$ be an irreducible variety of dimension n and let $f = (f_1, \dots, f_m) : X \to \mathbb{C}^m$ be a generically-finite polynomial mapping.

Then J_f is an algebraic subset of $\mathbb{C}^m$ and it is **either empty or it has pure dimension** $n-1$. *Moreover, if $n = m$ then*

$$\deg J_f \le \frac{\deg X (\prod_{i=1}^n \deg f_i) - \mu(f)}{\min_{1 \le i \le n} \deg f_i}$$

where $\mu(f) :=$ the number of points in a generic fiber of f.

$S \subset \mathbb{C}^n$ is called *horizontal* if $f(S)$ is not a point.
The union of all horizontal components of the polar curve $\Gamma(x_i, f_{i-1})$ will be called the *horizontal part* and will be denoted by $H\Gamma(x_i, f_{i-1})$.

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Theorem (Jelonek-T, 2017)

The set $NK_\infty(f)$ of non-trivial Malgrange non-regular values of f is included in the union of the non-properness sets of the mapping f restricted to the horizontal part of the polar curves $\Gamma(x_i, f_{i-1})$, more precisely we have the equality:

$$NK_\infty(f) = \bigcup_{i=1}^{n-1} J_f(H\Gamma(x_i, f_{i-1})) \setminus J_f(\mathrm{Sing} f). \tag{1}$$

Note that $J_f(\mathrm{Sing} f)$ is just the set of critical values of f which are images of fibers containing nonisolated singularities.

$\mathrm{Sing} f := S_0 \cup S_1 \cup \cdots \cup S_r$, the decomposition of the singular locus into irreducible components, where S_0 is the union of all point-components.

$d_i := \deg S_i$, the degree of the positive dimensional component S_i, for $i > 1$.

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Corollary

For $d > 2$ we have:

$$\#NK_\infty(f) \leq \frac{(d-1)^n - 1}{d-2} - \sum_{i=1}^{r} d_i \dim S_i, \tag{2}$$

and for $d = 2$:

$$\#NK_\infty(f) \leq n - 1 - \sum_{i=1}^{r} d_i \dim S_i.$$

Bound for the number of atypical values at infinity

Jelonek and Kurdyka (2014) found the following upper bound for Malgrange non-regular values:

$$\#K_\infty(f) \leq \frac{d^n - 1}{d + 1}. \tag{3}$$

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Both have the highest degree term d^{n-1}, and the coefficient of the term d^{n-2} in formula (4) is smaller than in formula (3) for high values of n.

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The first polar curve $\Gamma(x_1, f)$ detects some Malgrange non-regular value $c \in NK_\infty(f)$ whenever the fiber $f^{-1}(c)$ has only isolated t-singularities at infinity.

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For example, if $f(x, y, z) = x + x^2 y$ then the polar curve of f is empty, but f has a non-trivial Malgrange non-regular value 0.

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Is it possible to recover all non-trivial Malgrange non-regular values in just one single step?

Let us consider the following polynomials:

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$$g_i(a,b) = \sum_{j=1}^{n} a_{ij}\frac{\partial f}{\partial x_j} + \sum_{j,k=1}^{n} b_{ijk}x_k\frac{\partial f}{\partial x_j}, \ \ i = 1,\dots,n-1,$$

where a_{ij}, b_{ijk} are complex constants. Let:

$$\Gamma_f(a,b) := \mathrm{closure}\{V(g_1,\dots,g_{n-1}) \setminus \mathrm{Sing}(f)\}, \tag{5}$$

where we use here the Zariski closure. It turns out that, for general a_{ij}, b_{ijk} the set $\Gamma_f(a,b)$ is a non-empty curve, which we call *super-polar curve of* f.

Theorem (Jelonek-T, 2017)

The set $NK_\infty(f)$ *of nontrivial Malgrange non-regular values of* f *is included in the non-properness set of a mapping* f *restricted to the horizontal part of a sufficiently general super-polar curve* $\Gamma_f(a,b)$*, namely one has the following inclusion:*

$$NK_\infty(f) \subset J_f(H\Gamma_f(a,b)). \qquad (6)$$

Corollary

If $n > 2$ *and* $NK_\infty(f) \neq \emptyset$ *then:*

$$\#NK_\infty(f) \leq d^{n-1} - 1 - \sum_{i=1}^{r} d_i.$$

$$\#K_\infty(f) \leq d^{n-1} - 1 - \sum_{i=1}^{r} (d_i - 1).$$

Corollary

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$$\#NK_\infty(f) \leq d^{n-1} - 1$$