# On the postulation of lines and a fat line

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This talk is based on joint work (arXiv: 1706.02350) with

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and

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The slides are available at:
http://szpond.up.krakow.pl/MEGA2017.pdf

**Definition**

Let $I \subset R$ be a homogeneous ideal in a polynomial ring $R = \mathbb{K}[x_0, \ldots, x_N]$. The *Hilbert function* of $I$ is

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$$\mathrm{HF}_{R/I}(d) = \dim(R/I)_d.$$

**Remark**

*It is well-known that the Hilbert function becomes polynomial, i.e., there is a polynomial* $\mathrm{HP}_{R/I}(d)$ *such that*

$$\mathrm{HF}_{R/I}(d) = \mathrm{HP}_{R/I}(d) \quad \text{for} \quad d \gg 0.$$

**Remark**

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*If $I$ is a saturated ideal defining a subscheme $V \subset \mathbb{P}^N(\mathbb{K})$, then we write*

$$\mathrm{HF}_V(d) = \mathrm{HF}_{R/I}(d)$$

*and*

$$\mathrm{HP}_V(d) = \mathrm{HP}_{R/I}(d).$$

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*The simplest Hilbert functions occur for subvarieties which impose independent (or predictable) conditions on forms of arbitrary degree.*

**Definition (Carlini, Catalisano, Geramita)**

We say that a subscheme $V \subset \mathbb{P}^N(\mathbb{K})$ has a *bipolynomial Hilbert function* if

$$\mathrm{HF}_V(d) = \min\{\mathrm{HP}_{\mathbb{P}^N}(d),\ \mathrm{HP}_V(d)\}$$

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**Remark**

*There is*

$$\mathrm{HP}_{\mathbb{P}^N}(d) = \binom{N+d}{d}$$

*for all $d$.*

**Example**

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Let $V$ be a finite union of $s$ general points in a projective space $\mathbb{P}^N(\mathbb{K})$. Then $V$ has a bipolynomial Hilbert function.
More precisely we have

$$\mathrm{HF}_V(d) = \min\left\{ \binom{N+d}{d},\ s \right\}$$

for all $d$.

**Theorem (Alexander-Hirschowitz 1995)**

*Let $V$ be a general collection of $s$ double points in $\mathbb{P}^N(\mathbb{K})$ (over an algebraically closed field of characteristic zero). Then*

$$\mathrm{HF}_V(d) = \min\left\{ \binom{N+d}{d},\ s(N+1) \right\}$$

*except in the following cases*

- $d = 2, 2 \leq s \leq N$;
- $N = 2, d = 4, s = 5$;
- $N = 3, d = 4, s = 9$;
- $N = 4, d = 4, s = 14$;
- $N = 4, d = 3, s = 7$.

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**Remark**

*The authors worked on this problem for over 10 years.*

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### Remark

*All proofs are based on some* **degeneration***, i.e., if the claim holds for points in* **special** *position, then it holds for points in* **general** *position (provided both positions belong to a flat family).*

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# Passing from points to fat points, small values of $m$

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**Remark**

*Ciliberto and Miranda introduced a* **degeneration** *of the ambient space (replace $\mathbb{P}^2$ by some other scheme) combined with the degeneration of points.*

**Conjecture (SHGH, Segre-Harbourne-Gimigliano-Hirschowitz)**

*Let $V$ be a collection of $s$ general points of multiplicity $m$ in $\mathbb{P}^2(\mathbb{K})$. Then either*

$$\mathrm{HF}_V(d) = \min\left\{ \binom{d+2}{2}, \; s\binom{m+1}{2} \right\}$$

*or the linear system*

$$|\mathcal{O}_{\mathbb{P}^2}(d) \otimes \mathcal{I}_V^{(m)}|$$

*contains a fat $(-1)$-curve in its base locus.*

**Theorem (Hartshorne-Hirschowitz 1982)**

*Let $V$ be a union of $s$ general lines in the projective space $\mathbb{P}^N(\mathbb{K})$, with $N \geq 3$. Then the Hilbert function of $V$ is bipolynomial.*

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The hardest case is that of $N = 3$. The proof in $\mathbb{P}^3$ is based on a careful specialization of some lines onto a smooth quadric accompanied by a careful collision of some pairs of lines in points on the quadric.

**Theorem (Carlini, Catalisano, Geramita 2013, printed 2016)**

*Let $V$ be a union of $s$ general lines and a general point of multiplicity $m$ in the projective space $\mathbb{P}^N(\mathbb{K})$, with $N \geq 4$. Then the Hilbert function of $V$ is bipolynomial.*

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*More precisely we have*

$$\mathrm{HF}_V(d) = \min\left\{\binom{N+d}{d}, s(d+1) + \binom{m+N-1}{N}\right\}. \quad (1)$$

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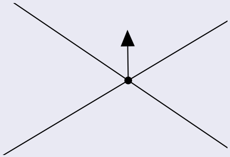
**Remark**

*In the case $N = 3$ the picture is more complicated:*
*The equality in (1) fails for*

$$2 \leq s \leq m \ \ and \ \ d = m.$$

Carlini, Catalisano and Geramita study again components of $V$ is special position and introduce **sundials**. These are degenerations of a pair of lines into a pair of intersecting lines with an embedded component in the intersection point.



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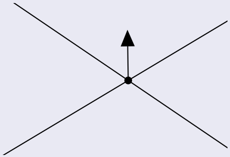

The case $N = 3$ is solved by Aladpoosh-Ballico (Rend. Sem. Mat. Univ. Pol. Torino 2015) and Ballico (Mediterranean Journal of Mathematics (2016)).

**Problem (Carlini, Catalisano, Geramita)**

*Identify Hilbert functions of subschemes in $\mathbb{P}^N$ consisting of the union of general lines and one fat linear subspace of arbitrary dimension.*

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*Identify Hilbert functions of subschemes in $\mathbb{P}^N$ consisting of the union of general lines and one fat linear subspace of arbitrary dimension.*
*Motivation: Determine the dimension of (higher) secant varieties to Segre embeddings of products of projective spaces.*

**Definition (Ballico)**

A "double" line $Y \subset \mathbb{P}^3$ is a connected divisor of type $(2,0)$ on a smooth quadric surface.

# (Half) fat lines

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**Remark**

*This generalizes a zero-dimensional subscheme of length 2 rather than a double point (which has length 3).*

**Theorem (Ballico 2012)**

*Let $V$ be a union of $s$ general lines and $t$ general "double" lines in $\mathbb{P}^3$. Then $V$ has a bipolynomial Hilbert function.*

**Definition**

A double line $X \subset \mathbb{P}^3$ is a subscheme supported on a line $L$ whose structure is determined by the square of the saturated ideal $I_L$ defining $L$.

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**Theorem (Aladpoosh 2016)**

*Let $V$ be a union of $s$ general lines and one general double line in $\mathbb{P}^N$, with $N \geq 3$. Then*

$$\mathrm{HF}_V(d) = \min \left\{ \binom{N+d}{d}, \ s(d+1) + (Nd+1) \right\}$$

*except in the case*

- $N = 4, \ s = 2, \ d = 2.$

**Theorem (Bauer, Di Rocco, Schmitz, Szemberg, Sz.)**

*Let $V$ be a union of $s$ general lines and one general line of multiplicity $m$ (i.e. defined by $I_L^m$) in $\mathbb{P}^3$. Then*

$$\mathrm{HF}_V(d) = \min\left\{\binom{d+3}{d},\ s(d+1) + \frac{1}{6}m(m+1)(3d+5-2m)\right\}$$

*for all $d \geq 3\binom{m+1}{3}$.*

**Definition (Zig-zag)**

A *zig-zag* of length $z$ is the limiting subscheme obtained by a collision of an ordered set of $z$ general lines $L_1, L_2, \dots, L_z$ in such a way, that the line $L_1$ intersects $L_2$, the line $L_2$ intersects $L_1$ and $L_3$ and the intersection points are distinct, $L_3$ intersects $L_2$ and $L_4$ and the intersection points are again distinct, and so on, finally $L_{z-1}$ intersects $L_{z-2}$ and $L_z$ in two distinct points. The structure in the intersection points is the same as the structure of a sundial in the intersection point of its lines.

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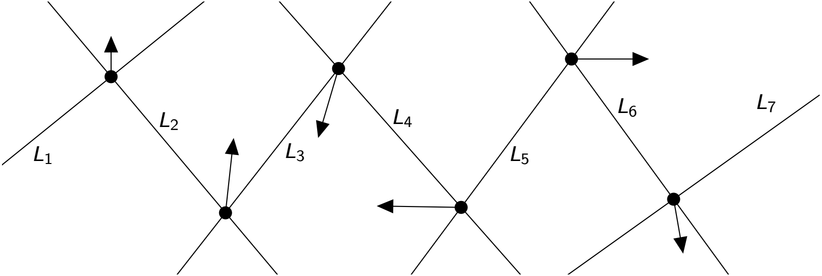
A zig-zag of length $z$ has thus $(z-1)$ singular points.
A sundial is a zig-zag of length 2.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Every second line of the **reduced** zig-zag is specialized on a smooth quadric as a general line, all in the same ruling.

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## Phase 2

Every second line of the **reduced** zig-zag is specialized on a smooth quadric as a general line, all in the same ruling. This quadric is also exhibited as a base component of the studied linear system. Removing it from the system decreases the degree again by 2. The residue of the reduced zig-zag is a collection of disjoint (general) lines.

**Definition (Trace and residual scheme)**

Let $Y$ be a smooth divisor in $\mathbb{P}^N$ and let $Z \subset \mathbb{P}^N$ be a closed subscheme. Then the subscheme $Z'' = \mathrm{Tr}_Y(Z)$ defined in $Y$ by the ideal

$$I_{Z''/Y} = (I_Y + I_Z)\,/I_Y \subset \mathcal{O}_Y$$

is the *trace of $Z$ on $Y$*.
The colon ideal $I_{Z'} = (I_Z : I_Y) \subset \mathcal{O}_{\mathbb{P}^N}$ defines $Z' = \mathrm{Res}_Y(Z)$, the *residual scheme of $Z$ with respect to $Y$*.

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We apply this sequence with $Y$ a smooth quadric in $\mathbb{P}^3$ and all terms twisted by $\mathcal{O}_{\mathbb{P}^3}(d)$.

## Lemma

*Let $Y \subset \mathbb{P}^N$ be a divisor of degree $e$ and let $d \geq e$ be an integer. Let $Z \subset \mathbb{P}^N$ be a closed subscheme. Then*

$$\begin{aligned} h^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(d) \otimes \mathcal{I}_Z) &\leq h^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(d-e) \otimes \mathcal{I}_{\mathrm{Res}_Y(Z)}) + \\ &+ h^0(Y, \mathcal{O}_Y(d) \otimes \mathcal{I}_{\mathrm{Tr}_Y(Z)/Y}). \end{aligned}$$

**Lemma**

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We call the space $H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(d-e) \otimes \mathcal{I}_{\mathrm{Res}_Y(Z)})$ the *residual linear system* of $H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(d) \otimes \mathcal{I}_Z)$ with respect to $Y$ and $H^0(Y, \mathcal{O}_Y(d) \otimes \mathcal{I}_{\mathrm{Tr}_Y(Z)/Y})$ the *trace linear system* of $H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(d) \otimes \mathcal{I}_Z)$ on $Y$.

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## Remark

*We have developed a computer software to*

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- **book-keeping!**

## Remark

*We have developed a computer software to*

- *make correct guesses on the bound on $d$;*
- *test various* **reduction steps***.*

# And that's it.

**Remark**

*The bound on $d$ given in the Main Theorem is due to the fact, that for $d$ big enough certain invariants of the ideal of $Z$ can be described by an explicit function. This makes the induction possible.*

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**Remark**

*We have checked by computer hundreds of cases and have found no irregularities in the Hilbert function.*

**Example**

Already in $\mathbb{P}^4$ some special cases come up. The easiest one is 1 double line $L$ and 2 ordinary lines $L_1$ and $L_2$. The union of two hypersurfaces generated by $L$ and $L_i$ vanishes double along $L$ and once along each $L_i$, whereas it is unexpected from the naive dimension count.

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**Remark**

*We expect however, that with a similar bound as in $\mathbb{P}^3$, the values of the Hilbert function in $\mathbb{P}^N$ can be computed by the bipolynomial formula.*

| For | a sequence of length | yields |
|---|---|---|
| $B(k,0,m)$ | 1 | $B(k-1,1,m-1)$ |
| $B(k,1,m)$ | 2 | $B(k-1,0,m-1)$ |
| $B(k,2,m)$ | 1 | $B(k,0,m-1)$ |
| $I(k,0,m)$ | 2 | $I(k-2,2,m-2)$ |
| $I(k,1,m)$ | 1 | $I(k-1,2,m-1)$ |
| $I(k,2,3\ell)$ | $3\ell-1$ | $B(k-2\ell+1,1,1)$ |
| $I(k,2,3\ell+1)$ | $3\ell+1$ | $B(k-2\ell,0,0)$ |
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