On conjectures of Chudnovsky and Demailly and Waldschmidt constants

Tomasz Szemberg
Pedagogical University of Cracow

MEGA 2017
Nice, June 16, 2017
Theorem

Let $f_1, \ldots, f_k$ be meromorphic functions in $\mathbb{C}$ with $f_1, f_2$ algebraically independent. Let $\mathbb{K}$ be a number field. Assume that for all $j = 1, \ldots, k$

$$f_j' \in \mathbb{K}[f_1, \ldots, f_k].$$

Then the set

$$S = \{ z \in \mathbb{C} : z \text{ is not a pole of } f_j, f_j(z) \in \mathbb{K}, j = 1, \ldots, k \}$$

is finite.
**Theorem**

Let $f_1, \ldots, f_k$ be meromorphic functions in $\mathbb{C}$ with $f_1, f_2$ algebraically independent. Let $\mathbb{K}$ be a number field. Assume that for all $j = 1, \ldots, k$

$$f_j' \in \mathbb{K}[f_1, \ldots, f_k].$$

Then the set

$$S = \{z \in \mathbb{C} : z \text{ is not a pole of } f_j, f_j(z) \in \mathbb{K}, j = 1, \ldots, k\}$$

is finite.

**Corollary (Hermite-Lindemann)**

For $\omega \in \mathbb{C}^*$ at least one of the numbers $\omega, \exp(\omega)$ is transcendental.
Schwarz Lemma in one variable

**Theorem**

Let $f$ be an analytic function in a disc $\{ |z| \leq R \} \subset \mathbb{C}$ with at least $N$ zeroes in a disc $\{ |z| \leq r \}$ with $r < R$. Then

$$|f|_r \leq \left( \frac{3r}{R} \right)^N |f|_R,$$

where

$$|f|_\gamma = \sup_{|z| \leq \gamma} |f(z)|.$$
Theorem (Bombieri 1970)

Let $f_1, \ldots, f_k$ be meromorphic functions in $\mathbb{C}^n$ with $f_1, \ldots, f_{n+1}$ algebraically independent. Let $\mathbb{K}$ be a number field. Assume that for all $i = 1, \ldots, n$, $j = 1, \ldots, k$

$$\frac{\partial}{\partial z_i} f_j \in \mathbb{K}[f_1, \ldots, f_k].$$

Then the set

$$S = \{z \in \mathbb{C}^n : z \text{ is not a pole of } f_j, f_j(z) \in \mathbb{K}, j = 1, \ldots, k\}$$

is contained in an algebraic hypersurface.
Theorem

Let $S \subset \mathbb{C}^n$ be a finite set. Let $m$ be a positive integer. There exists $M(m) > 0$ such that there exists $r > 0$ such that for $R > r$ and a function $f$ analytic in the ball $\{|z| \leq R\} \subset \mathbb{C}^n$ vanishing with multiplicity $\geq m$ at each point of $S$

$$|f|_r \leq \left( \frac{c(n) \cdot r}{R} \right)^{M(m)} |f|_R,$$

where $c(n)$ is a constant depending only on $n$. 

Problem

Make the statement effective. In particular: what is the maximal value of $M(m)$?
Theorem

Let $S \subset \mathbb{C}^n$ be a finite set. Let $m$ be a positive integer. There exists $M(m) > 0$ such that there exists $r > 0$ such that for $R > r$ and a function $f$ analytic in the ball $\{|z| \leq R\} \subset \mathbb{C}^n$ vanishing with multiplicity $\geq m$ at each point of $S$

$$|f|_r \leq \left( \frac{c(n) \cdot r}{R} \right)^{M(m)} |f|_R,$$

where $c(n)$ is a constant depending only on $n$.

Problem

Make the statement effective. In particular: what is the maximal value of $M(m)$?
Waldschmidt constant should be Moreau constant

**Problem**

*Make the statement effective. In particular: what is the maximal value of $M$?*

**Theorem (Moreau)**

Let $S \subset \mathbb{C}^n$ be a finite set. Let $m$ be a positive integer. There exists $r > 0$ such that for $R > r$ and a function $f$ analytic in the ball $\{ |z| \leq R \} \subset \mathbb{C}^n$ vanishing with multiplicity $\geq m$ at each point of $S$,

$$|f|_r \leq (\exp(n) \cdot r R)^{\alpha(mS)} |f|R,$$

where $\alpha(mS)$ is the initial degree of $I(mS)$. 

**Remark**
The constant $\alpha(mS)$ is optimal.
Problem

Make the statement effective. In particular: what is the maximal value of $M$?

Theorem (Moreau)

Let $S \subset \mathbb{C}^n$ be a finite set. Let $m$ be a positive integer. There exists $r > 0$ such that for $R > r$ and a function $f$ analytic in the ball $\{ |z| \leq R \} \subset \mathbb{C}^n$ vanishing with multiplicity $\geq m$ at each point of $S$

$$|f|_r \leq \left( \frac{\exp(n) \cdot r}{R} \right)^{\alpha(mS)} |f|_R,$$

where $\alpha(mS)$ is the initial degree of $I_S^{(m)}$. 
Problem

Make the statement effective. In particular: what is the maximal value of $M$?

Theorem (Moreau)

Let $S \subset \mathbb{C}^n$ be a finite set. Let $m$ be a positive integer. There exists $r > 0$ such that for $R > r$ and a function $f$ analytic in the ball $\{|z| \leq R\} \subset \mathbb{C}^n$ vanishing with multiplicity $\geq m$ at each point of $S$

$$|f|^r \leq \left(\frac{\exp(n) \cdot r}{R}\right)^{\alpha(mS)} |f|^R,$$

where $\alpha(mS)$ is the initial degree of $I_S^{(m)}$.

Remark

The constant $\alpha(mS)$ is optimal.
Symbolic powers

Definition

Let $\mathbb{K}$ be a field and let $R = \mathbb{K}[x_0, \ldots, x_n]$ be the ring of polynomials. For a homogeneous ideal $0 \neq I \subset R$ its $m$-th symbolic power is

$$I^{(m)} = \bigcap_{P \in \text{Ass}(I)} (I^m R_P \cap R).$$
Symbolic powers

Definition

Let $K$ be a field and let $R = K[x_0, \ldots, x_n]$ be the ring of polynomials. For a homogeneous ideal $0 \neq I \subset R$ its $m$-th symbolic power is

$$I^{(m)} = \bigcap_{P \in \text{Ass}(I)} (I^m R_P \cap R).$$

Theorem (Zariski-Nagata)

Let $X \subset \mathbb{P}^n(K)$ be a projective variety (in particular reduced). Then $I(X)^{(m)}$ is generated by all forms which vanish along $X$ to order at least $m$. 

Tomasz Szemberg Pedagogical University of Cracow

Waldschmidt constants
Let $Z = \{P_1, \ldots, P_s\}$ be a finite set of points in $\mathbb{P}^n(K)$. Then

$$I(Z) = I(P_1) \cap \ldots \cap I(P_s)$$

and

$$I(Z)^{(m)} = I(P_1)^m \cap \ldots \cap I(P_s)^m$$

for all $m \geq 1$. 
Definition

For a graded ideal \( I \) its *initial degree* \( \alpha(I) \) is the least number \( t \) such that \( I_t \neq 0 \).
The initial degree and the Waldschmidt constant

**Definition**

For a graded ideal $I$ its *initial degree* $\alpha(I)$ is the least number $t$ such that $I^t \neq 0$. The *Waldschmidt constant* of $I$ is the real number

$$\hat{\alpha}(I) = \inf_{m \geq 1} \frac{\alpha(I^{(m)})}{m}.$$
Conjecture (Nagata)

Let $I$ be a saturated ideal of $s \geq 10$ very general points in $\mathbb{P}^2(\mathbb{C})$. Then

$$\alpha(I^{(m)}) > m\sqrt{s}$$

for all $m \geq 1$. 

Waldschmidt constants are hard to compute
Conjecture (Nagata)

Let $I$ be a saturated ideal of $s \geq 10$ very general points in $\mathbb{P}^2(\mathbb{C})$. Then

$$\alpha(I^{(m)}) > m\sqrt{s}$$

for all $m \geq 1$.

Equivalently

$$\hat{\alpha}(I) = \sqrt{s}.$$
Conjecture (Chudnovsky)

Let $I$ be a saturated ideal of points in $\mathbb{P}^n(\mathbb{K})$. Then

$$\hat{\alpha}(I) \geq \frac{\alpha(I) + n - 1}{n}.$$
Conjecture (Chudnovsky)

Let $I$ be a saturated ideal of points in $\mathbb{P}^n(K)$. Then

$$\hat{\alpha}(I) \geq \frac{\alpha(I) + n - 1}{n}.$$ 

Conjecture (Demailly)

Let $I$ be a saturated ideal of points in $\mathbb{P}^n(K)$. Then

$$\hat{\alpha}(I) \geq \frac{\alpha(I^{(m)}) + n - 1}{m + n - 1}.$$
Conjecture (Chudnovsky)

Let \( I \) be a saturated ideal of points in \( \mathbb{P}^n(K) \). Then

\[
\hat{\alpha}(I) \geq \frac{\alpha(I) + n - 1}{1 + n - 1}.
\]

Conjecture (Demailly)

Let \( I \) be a saturated ideal of points in \( \mathbb{P}^n(K) \). Then

\[
\hat{\alpha}(I) \geq \frac{\alpha(I^{(m)}) + n - 1}{m + n - 1}.
\]

Remark

The Chudnovsky Conjecture is the \( m = 1 \) case of the Demailly Conjecture.
Problem

*Compare ordinary and symbolic powers of homogeneous ideals.*
The containment problem

Problem

Compare ordinary and symbolic powers of homogeneous ideals. More precisely, given $I$ determine all pairs $(m, r)$ such that
a) $I^r \subset I^{(m)}$;
b) $I^{(m)} \subset I^r$.

Proposition

$I^r \subset I^{(m)} \iff r \geq m$.

Theorem (Ein-Lazarsfeld-Smith, Hochster-Huneke)

If $m \geq \dim(I) + r$, then $I^{(m)} \subset I^r$. 

Tomasz Szemberg Pedagogical University of Cracow

Waldschmidt constants
The containment problem

Problem

*Compare ordinary and symbolic powers of homogeneous ideals. More precisely, given I determine all pairs \((m, r)\) such that*

\[a) \ I^r \subset I^{(m)}; \]

\[b) \ I^{(m)} \subset I^r. \]

Proposition

\[I^r \subset I^{(m)} \iff r \geq m. \]
The containment problem

**Problem**

*Compare ordinary and symbolic powers of homogeneous ideals. More precisely, given I determine all pairs \((m, r)\) such that*

\[ I^r \subset I^{(m)}; \]

\[ I^{(m)} \subset I^r. \]

**Proposition**

\[ I^r \subset I^{(m)} \iff r \geq m. \]

**Theorem (Ein-Lazarsfeld-Smith, Hochster-Huneke)**

*If \( m \geq \text{height}(I) r \), then \( I^{(m)} \subset I^r \).*
Containment and Chudnovsky Conjecture

**Theorem (Ein, Lazarsfeld, Smith; Hochster, Huneke)**

Let $I$ be a saturated ideal in $\mathbb{K}[x_0, \ldots, x_n]$. Then for all $m \geq nr$

$$I^{(m)} \subset I^r.$$
Theorem (Ein, Lazarsfeld, Smith; Hochster, Huneke)

Let $I$ be a saturated ideal in $\mathbb{K}[x_0, \ldots, x_n]$. Then for all $m \geq nr$

$$I^{(m)} \subset I^r.$$ 

Corollary

Let $I$ be a saturated ideal of points in $\mathbb{P}^n(\mathbb{K})$. Then

$$\hat{\alpha}(I) \geq \frac{\alpha(I)}{n}.$$
Theorem (Ein, Lazarsfeld, Smith; Hochster, Huneke)

Let $I$ be a saturated ideal in $\mathbb{K}[x_0, \ldots, x_n]$. Then for all $m \geq nr$

$$I^{(m)} \subset I^r.$$ 

Corollary

Let $I$ be a saturated ideal of points in $\mathbb{P}^n(\mathbb{K})$. Then

$$\hat{\alpha}(I) \geq \frac{\alpha(I)}{n}.$$ 

Theorem (Demailly)

In the set up as above

$$\hat{\alpha}(I) \geq \frac{\alpha(I)(\alpha(I) + 1) \cdot \ldots \cdot (\alpha(I) + n - 1)}{n! \alpha(I)^{n-1}}.$$
Definition (Star configuration of points)

We say that $Z \subset \mathbb{P}^N$ is a *star configuration* of degree $d$ (or a $d$-star for short) if $Z$ consists of all intersection points of $d \geq N$ general hyperplanes in $\mathbb{P}^N$. By intersection points we mean the points which belong to exactly $N$ of given $d$ hyperplanes.
Definition (Star configuration of points)

We say that $Z \subset \mathbb{P}^N$ is a star configuration of degree $d$ (or a $d$-star for short) if $Z$ consists of all intersection points of $d \geq N$ general hyperplanes in $\mathbb{P}^N$. By intersection points we mean the points which belong to exactly $N$ of given $d$ hyperplanes.

Example (Bocci, Harbourne)

For points in the star configuration in $\mathbb{P}^n$, there is the equality in the Chudnovsky Conjecture.
Theorem (Bocci, Cooper, Guardo, Harbourne, Janssen, Nagel, Seceleanu, Van Tuyl, Vu 2015)

Let $I$ be a squarefree monomial ideal with $\text{bight}(I) = e$. Then

$$\hat{\alpha}(I) \geq \frac{\alpha(I) + e - 1}{e}.$$
Theorem (Esnault – Viehweg 1983)

Let \( I \) be a radical ideal of a finite set of points in \( \mathbb{P}^n \) with \( n \geq 2 \). Let \( k \leq m \) be two integers. Then

\[
\frac{\alpha(I^{(k)}) + 1}{k + n - 1} \leq \frac{\alpha(I^{(m)})}{m},
\]

in particular

\[
\frac{\alpha(I^{(k)}) + 1}{k + n - 1} \leq \hat{\alpha}(I).
\]
More evidence for the Chudnovsky Conjecture

**Theorem (Esnault – Viehweg 1983)**

*Let $I$ be a radical ideal of a finite set of points in $\mathbb{P}^n$ with $n \geq 2$. Let $k \leq m$ be two integers. Then*

$$\frac{\alpha(I^{(k)}) + 1}{k + n - 1} \leq \frac{\alpha(I^{(m)})}{m},$$

*in particular*

$$\frac{\alpha(I^{(k)}) + 1}{k + n - 1} \leq \hat{\alpha}(I).$$

**Corollary**

*In particular*

$$\frac{\alpha(I) + 1}{n} \leq \hat{\alpha}(I).$$

*Hence the Chudnovsky Conjecture holds in $\mathbb{P}^2$.***
General points

Theorem (Dumnicki-Tutaj-Gasińska, Fouli-Manteo-Xie 2016)

*The Chudnovsky Conjecture holds for general points in $\mathbb{P}^n$.***
General points

Theorem (Dumnicki-Tutaj-Gasińska, Fouli-Manteo-Xie 2016)

The Chudnovsky Conjecture holds for general points in $\mathbb{P}^n$.

Theorem (Malara, Szemberg, Szpond 2017)

The Demailly Conjecture holds for general points in $\mathbb{P}^n$. 
Theorem (Dumnicki, Szemberg, Szpond 2017)

Let $H_1, \ldots, H_s$ be $s \geq 2$ mutually distinct hyperplanes in $\mathbb{P}^N$. Let $a_1 \geq \ldots \geq a_s > 1$ be real numbers such that

$$
\begin{align*}
& a_1 - 1 > 0 \\
& a_1a_2 - a_1 - a_2 > 0 \\
& \vdots \\
& a_1 \ldots a_{s-1} - \sum_{i=1}^{s-1} a_1 \ldots \hat{a}_i \ldots a_{s-1} > 0 \\
& a_1 \ldots a_s - \sum_{i=1}^{s} a_1 \ldots \hat{a}_i \ldots a_s \leq 0.
\end{align*}
$$

For $i = 1, \ldots, s$ let

$$Z_i = \{P_{i,1}, \ldots, P_{i,k_i}\} \in H_i \setminus \bigcup_{j \neq i} H_j$$

be a set of $k_i$ points such that $\hat{\alpha}(H_i; Z_i) \geq a_i$. 
Theorem (Dumnicki, Szemberg, Szpond 2017)

For \( Z = \bigcup_{i=1}^{s} Z_i \) and

\[
q := \frac{a_1 \ldots a_{s-1} - \sum_{i=1}^{s-1} a_1 \ldots \hat{a}_i \ldots a_{s-1}}{a_1 \ldots a_{s-1} \cdot a_s + s - 1}.
\]

There is

\[\hat{\alpha}(\mathbb{P}^N; Z) \geq q.\]