Macaulay style formulae for the sparse resultant

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Méthodes Effectives en Géométrie Algébrique (MEGA 2017), Nice, June 2017
Given

\[
\begin{align*}
  c_{0,0}x_0 &+ \cdots + c_{0,n}x_n = 0 \\
  \vdots & \quad \vdots \\
  c_{n,0}x_0 &+ \cdots + c_{n,n}x_n = 0
\end{align*}
\]

the condition that this linear system has a nontrivial solution is

\[
\text{det}(c_{i,j})_{i,j} = 0
\]
The Sylvester resultant

Let

\[ F = a_0 x_0^m + a_1 x_0^{m-1} x_1 + a_2 x_0^{m-2} x_1^2 + \ldots + a_m x_1^m \]

\[ G = b_0 x_0^k + b_1 x_0^{k-1} x_1 + b_2 x_0^{k-2} x_1^2 + \ldots + b_k x_1^k \]

Their **Sylvester resultant** is the irreducible polynomial in \( \mathbb{Z}[a_0, \ldots, a_m, b_0, \ldots, b_k] \) given by

\[
\text{Res}_{m,k}(F, G) = \det \begin{pmatrix}
  a_m & a_{m-1} & \cdots & a_0 \\
  a_m & a_{m-1} & \cdots & a_0 \\
  \vdots & \vdots & \ddots & \vdots \\
  b_k & b_{k-1} & \cdots & b_0 \\
  b_k & b_{k-1} & \cdots & b_0 \\
  \vdots & \vdots & \ddots & \vdots \\
  b_k & b_{k-1} & \cdots & b_0
\end{pmatrix}
\]

Vanishes iff \( \exists \xi \in \mathbb{P}^1 \) such that \( F(\xi) = G(\xi) = 0 \)
The multivariate resultant

For $i = 0, \ldots, n$ let

$$F_i = \sum_{a_0 + \cdots + a_n = d_i} c_i, a \cdot x_0^{a_0} \cdots x_n^{a_n}$$

homogeneous polynomial in the variables $x_0, x_1, \ldots, x_n$ of degree $d_i$

Set $c_i := (c_i, a)_{|a|=d_i}$. The multivariate resultant

$$\text{Res}_{d_0, \ldots, d_n}(F_0, \ldots, F_n) \in \mathbb{Z}[c_0, \ldots, c_n]$$

is the unique irreducible polynomial that vanishes iff $\exists \xi \in \mathbb{P}^n$ st

$$F_1(\xi) = \cdots = F_n(\xi) = 0$$
Properties

- $\text{Res}_{d_0, d_1, \ldots, d_n}(F_0, \ldots, F_n)$ is homogeneous in the variables $c_j$ of degree $\prod_{j \neq i} d_j$ for each $j$
- Weighted homogeneous of degree $\prod_j d_j$

Satisfies Poisson’s formula:

$$\text{Res}_{d_0, d_1, \ldots, d_n}(F_0, F_1, \ldots, F_n) = \text{Res}_{d_1, \ldots, d_n}(F_1^\infty, \ldots, F_n^\infty)^{d_0} \prod_{F_1(\xi) = \cdots = F_n(\xi) = 0} \frac{F_0(\xi)}{\xi^{d_0}}$$
A known case: if \( d_i = 1 \) for all \( i \), then

\[
\text{Res}_{d_0, \ldots, d_n}(F_0, \ldots, F_n) = \det(c_{i,j})_{i,j}
\]

A less trivial case:

\[
F_0 = c_{0,0}x_0 + c_{0,1}x_1 + c_{0,2}x_2 \\
F_1 = c_{1,0}x_0 + c_{1,1}x_1 + c_{1,2}x_2 \\
F_2 = c_{2,0}x_0^2 + c_{2,1}x_0x_1 + c_{2,2}x_0x_2 + c_{2,3}x_1^2 + c_{2,4}x_1x_2 + c_{2,5}x_2^2
\]

then \( \text{Res}_{1,1,2}(F_0, F_1, F_2) \) is

\[
\begin{align*}
&c_{0,0}^2 c_{1,1}^2 c_{2,5}^2 c_{0,0} c_{1,1} c_{1,2} c_{2,4} + c_{0,0}^2 c_{1,2}^2 c_{2,3}^2 c_{0,0} c_{0,1} c_{1,0} c_{1,1} c_{2,5} + c_{0,0} c_{0,1} c_{1,0} c_{1,1} c_{2,5} + c_{0,0} c_{0,1} c_{1,0} c_{1,2} c_{2,4} \\
&+ c_{0,0} c_{0,1} c_{1,1} c_{1,2} c_{2,2}^2 - c_{0,0} c_{0,1} c_{1,2} c_{2,1}^2 c_{1,0} c_{1,1} c_{2,4} c_{0,0} c_{0,2} c_{1,0} c_{1,1} c_{2,2} + c_{0,0} c_{0,1} c_{1,0} c_{1,1} c_{2,2} c_{2,3} \\
&- c_{0,0} c_{0,2} c_{1,1} c_{2,2} + c_{0,0} c_{0,2} c_{1,1} c_{1,2} c_{2,1} + c_{0,0}^2 c_{1,1} c_{1,0} c_{2,5} c_{2,2} c_{0,1} c_{1,0} c_{1,1} c_{1,2} c_{2,2} + c_{0,1}^2 c_{1,2}^2 c_{2,0} \\
&- c_{0,1} c_{0,2} c_{1,0} c_{2,4} + c_{0,1} c_{0,2} c_{1,0} c_{1,1} c_{2,2} c_{1,0} c_{1,2} c_{2,1} - c_{0,1} c_{0,2} c_{1,1} c_{2,1} c_{2,0} \\
&+ c_{0,2} c_{1,0} c_{2,3} c_{0,2} c_{1,2} c_{1,1} c_{2,1} + c_{0,2}^2 c_{1,1} c_{2,0}
\end{align*}
\]
The Macaulay formula (1916)

\[ \text{Res}_{d_0, \ldots, d_n} = \frac{\det(M)}{\det(E)} \]

with $M$ a "Sylvester" matrix and $E$ a block diagonal submatrix.
The general elimination problem

Given a subvariety

\[ \Omega \subset X \times Y \]

compute (= give equations for) the image

\[ \text{pr}_2(\Omega) \subset Y \]
Let $\mathbb{C} \rightarrow \mathbb{C}^2$ a rational map given by

$$t \mapsto \left( \frac{p(t)}{r(t)}, \frac{q(t)}{s(t)} \right)$$

with $p, q, r, s \in \mathbb{C}[t]$ such that $\gcd(p, r) = 1$ and $\gcd(q, s) = 1$

The *implicit equation* is

$$E(x, y) = \text{Res}^t (r(t)x - p(t), s(t)y - q(t))$$

**Example.** The implicit equation of the image of

$$t \mapsto \left( \frac{3t}{1+t^3}, \frac{3t^2}{1+t^3} \right)$$

is

$$x^3 + y^3 - 3xy = 0$$
Flexes of hypersurfaces

A point $\xi$ of a surface $S \subset \mathbb{P}^3$ is a flex if there is a line $L$ with order of contact $\geq 4$ at $\xi$. 
Theorem (Salmon 1862)

If $S$ is not ruled of degree $D$, then $\text{Flex}(S)$ is an algebraic curve of degree

$$\leq D \cdot (11D - 24)$$

For $D = 3$, $\deg(\text{Flex}(S)) \leq 27$
Distinct distances

Conjecture (Erdős 1946)

$n$ points in the plane define at least $\Omega\left(\frac{n}{\sqrt{\log n}}\right)$ distinct distances

Guth & Katz (Ann. Math. 2015) proved that they define at least $\Omega\left(\frac{n}{\log n}\right)$ such distances
A theorem on incidences

Their proof realizes the Elekes’ program, that reduces Erdös’ conjecture to a problem on incidences

**Theorem (Guth & Katz 2015)**

Let $\mathcal{L}$ be a set of $n^2$ lines in $\mathbb{R}^3$ with at most $n$ of them lying in a doubly ruled surface. For $k \leq n$, the number of points in a $k$ of the lines in $\mathcal{L}$ is bounded by

$$O\left(\frac{n^3}{k^2}\right)$$

Proven using the *polynomial partitioning method* and Salmon’s theorem on flexes
A point $\xi$ in a hypersurface $S \subset \mathbb{P}^n$ is a flex if there is a line $L$ with order of contact $\geq n + 1$ at $\xi$

Let $F \in \mathbb{C}[x_0, \ldots, x_n]$ homogeneous of degree $D$ st $S = V(F)$. Write

$$F(x + ty) = \sum_{i=0}^{n} F_i(x, y) t^i + O(t^{n+1})$$

Then $\xi \in \text{Flex}(S)$ iff $\exists \eta \neq \xi$ st

$$F_0(\xi, \eta) = \cdots = F_n(\xi, \eta) = 0$$

and $\text{Flex}(S)$ is defined by

$$F = \text{Res}_{1,2,3,\ldots,n}^y (F_0^\infty, \ldots, F_n^\infty) = 0$$
Theorem

\[ \exists H \in \mathbb{C}[x_0, \ldots, x_n] \text{ such that } \]
\[ \text{Res}_{1,2,3,\ldots,n}(F_0^\infty, \ldots, F_n^\infty) = x_0^nH \pmod{F} \]

Hence

\[ \text{Flex}(S) = V(F, H) \]

By Bézout, if \( S \) not ruled, then Flex(\( S \)) is a codimension 1 subvariety of \( S \) of degree

\[ \leq D \cdot \left( \left( \sum_{i=1}^{n} \frac{n!}{i} \right) D - (n + 1)! \right) \]

When \( F \) is generic, this bound is an equality
Not the end of the story?

When $n = 2$ we can take

$$H = \text{Hess}(f) = \det \begin{pmatrix} \frac{\partial^2 F}{\partial x_0^2} & \frac{\partial^2 F}{\partial x_0 \partial x_1} & \frac{\partial^2 F}{\partial x_0 \partial x_2} \\ \frac{\partial^2 F}{\partial x_0 \partial x_1} & \frac{\partial^2 F}{\partial x_1^2} & \frac{\partial^2 F}{\partial x_1 \partial x_2} \\ \frac{\partial^2 F}{\partial x_0 \partial x_2} & \frac{\partial^2 F}{\partial x_1 \partial x_2} & \frac{\partial^2 F}{\partial x_2^2} \end{pmatrix}$$

What about $n \geq 3$?
For $i = 0, \ldots, n$ let $\mathcal{A}_i \subset \mathbb{Z}^n$ be a finite set and

$$f_i = \sum_{a \in \mathcal{A}_i} c_{i,a} x^a$$

a Laurent polynomial in $x = (x_1, \ldots, x_n)$ with exponents in $\mathcal{A}_i$.

Set $\mathcal{A} = (\mathcal{A}_0, \ldots, \mathcal{A}_n)$

What is $\text{Res}_\mathcal{A}$?
One issue

Incidence subvariety:

\[ \Omega_{\mathcal{A}} = \{ (x, c_0, \ldots, c_n) \mid f_i(x) = 0 \; \forall i \} \]

Projection:

\[ \pi : (\mathbb{C}^\times)^n \times \prod_{i=0}^{n} \mathbb{P}(\mathbb{C}^{A_i}) \to \prod_{i=0}^{n} \mathbb{P}(\mathbb{C}^{A_i}) \]

Example. \( A_0 = A_1 = A_2 = \{ (0, 0), (1, 0) \} \subset \mathbb{Z}^2 \). Then

\[ f_0 = c_{0,0} + c_{0,1}x_1, \; f_1 = c_{1,0} + c_{1,1}x_1, \; f_2 = c_{2,0} + c_{2,1}x_1 \]

and \( \pi(\Omega_{\mathcal{A}}) \) not of codimension 1
For \( I \subset \{0, \ldots, n\} \) set \( \mathcal{A}_I = (A_i)_{i \in I} \) and \( L_{\mathcal{A}_I} = \sum_{i \in I} L_{A_i} \) with
\[
L_{A_i} = \mathbb{Z} \cdot (A_i - A_i) \subset \mathbb{Z}^n
\]

\( \mathcal{A}_I \) is essential if
- \( \# I = \text{rank}(L_{\mathcal{A}_I}) + 1 \)
- \( \# I' \leq \text{rank}(L_{\mathcal{A}_{I'}}) \) for all \( I' \subsetneq I \).

**Fact.** (Sturmfels) \( \text{codim} \pi(\Omega_{\mathcal{A}}) = 1 \) iff \( \exists! \) essential subfamily of \( \mathcal{A} \)

**Definition** (Gelfand-Kapranov-Zelevinski 1994, Sturmfels 1994)

\( \text{Elim}_{\mathcal{A}} \) is the irreducible polynomial in \( \mathbb{Z}[c_0, \ldots, c_n] \) giving an equation for \( \pi(\Omega_{\mathcal{A}}) \), if it is a hypersurface, and \( \text{Elim}_{\mathcal{A}} = 1 \) otherwise.
Another issue

Example. $A_0 = A_1 = \{0, 2\} \subset \mathbb{Z}$. Then

$$f_0 = c_{0,0} + c_{0,2}x^2, \quad f_1 = c_{1,0} + c_{1,2}x^2$$

$\pi(\Omega_A)$ has codimension 1 but $\pi|_{\Omega_A}$ not birational

Definition (Esterov 2014, D’Andrea-S 2015)

$\text{Res}_A$ is the primitive polynomial in $\mathbb{Z}[c_0, \ldots, c_n]$ giving an equation for $\pi_*\Omega_A$

Hence

$$\text{Res}_A = \text{Elim}_A^{\deg(\pi|_{\Omega_A})}$$

and $\deg(\pi|_{\Omega_A})$ can be computed by a (complicated) combinatorial expression
Properties of the $\mathcal{A}$-resultant

Set $Q_i = \text{conv}(\mathcal{A}_i) \subset \mathbb{R}^n$

$\text{Res}_\mathcal{A}$ homogeneous in the variables $c_i$ of degree

$$\text{MV}(Q_0, \ldots, Q_{i-1}, Q_{i+1}, \ldots, Q_n)$$

Poisson’s formula (D’Andrea-S 2015)

$$\text{Res}_\mathcal{A}(f_0, \ldots, f_n) = \prod_{v \in \mathbb{Z}^n \text{ primitive}} \text{Res}_{\overline{\mathcal{A}}_v}(f_1,v, \ldots, f_n,v)^{-h_{\mathcal{A}_0}(v)} \prod_{\xi \in V(f_1, \ldots, f_n)} f_0(\xi)$$

with

- $\overline{\mathcal{A}}_v = (\mathcal{A}_1,v, \ldots, \mathcal{A}_n,v)$, $\mathcal{A}_i,v = \{a \in \mathcal{A}_i \mid \langle a, v \rangle \text{ minimum} \}$
- $h_{\mathcal{A}_0}(v) = \min\{\langle a, v \rangle \mid a \in \mathcal{A}_0 \}$
More properties and formulae

Joint with C. D’Andrea and G. Jeronimo

For \( i = 0, \ldots, n \) let \( \omega_i \in \mathbb{R}^{A_i} \) and consider the lifted polytope

\[
Q_i,\omega_i = \text{conv}(\{(a, \omega_i a) \mid a \in A_i\}) \subset \mathbb{R}^{n+1}
\]

For \( \mathbf{v} \in \mathbb{Z}^{n+1} \) let \( A_{i,v} \subset A_i \) the part of minimal \( \mathbf{v} \)-weight and

\[
f_{i,v} = \sum_{a \in A_{i,v}} c_{i,a} x^a
\]

the “restriction” of \( f_i \) to \( A_{i,v} \). Set \( \omega = (\omega_0, \ldots, \omega_n) \)

### Theorem

\[
\text{init}_\omega(\text{Res}_A) = \prod_{\mathbf{v}} \text{Res}_{A_0,v,\ldots,A_n,v}(f_{0,v} \ldots, f_{n,v})
\]

product over all \( \mathbf{v} \in \mathbb{Z}^{n+1} \) primitive inner normals to the facets of the lower envelope of \( Q_{0,\omega_0} + \cdots + Q_{n,\omega_n} \).
Example

\[ \mathcal{A}_0 = \{(0,0),(1,3),(2,2)\}, \quad \mathcal{A}_1 = \{(0,0),(1,2),(2,0)\}, \quad \mathcal{A}_2 = \{(1,1),(3,0)\} \]

Then

\[ \text{Res}_A = -u_{1,12} u_{1,00} u_{0,22} u_{0,13}^2 u_{1,20}^5 u_{1,21}^2 u_{2,30}^2 u_{0,00}^2 + 3 u_{1,12}^3 u_{0,22}^2 u_{1,20}^2 u_{2,11}^2 u_{2,30}^2 u_{0,00}^2 \]
\[ + 5 u_{1,12}^3 u_{1,00}^2 u_{0,13}^2 u_{0,22} u_{2,11}^2 u_{2,30}^2 u_{0,00}^2 - 7 u_{1,12}^5 u_{1,00}^3 u_{0,13}^2 u_{1,20} u_{2,11} u_{2,30}^2 u_{0,00}^2 \]
\[ + 2 u_{1,12}^4 u_{1,00}^2 u_{0,13}^2 u_{0,22} u_{1,20} u_{2,11} u_{2,30} u_{0,00}^2 \]
\[ - 2 u_{1,12}^5 u_{1,00}^3 u_{0,13}^2 u_{0,22} u_{1,20} u_{2,11} u_{2,30} u_{0,00}^2 \]
\[ + u_{1,12}^7 u_{2,11} u_{2,30} u_{0,00}^2 - 13 u_{0,13}^2 u_{0,22} u_{1,00}^2 u_{1,12} u_{1,20} u_{2,11} u_{2,30}^2 u_{0,00}^2 \]
\[ - 2 u_{0,13}^3 u_{0,22} u_{1,00}^2 u_{1,20} u_{2,11} u_{2,30}^2 u_{0,00}^2 + u_{1,12}^5 u_{1,00}^3 u_{0,22} u_{2,11} u_{2,30}^2 u_{0,00}^2 \]
\[ + 6 u_{1,12}^4 u_{1,00}^2 u_{0,22} u_{1,20} u_{2,11} u_{2,30} u_{0,00}^2 - 7 u_{1,12}^4 u_{1,00}^2 u_{0,13}^2 u_{1,20} u_{2,11} u_{2,30}^2 u_{0,00}^2 \]
\[ + u_{2,30}^3 u_{1,00}^5 u_{0,13}^2 \]
\[ + u_{1,12}^5 u_{0,22} u_{1,00}^2 u_{1,20} u_{2,11} u_{2,30}^2 u_{0,00}^2 - 5 u_{0,13}^2 u_{0,22} u_{1,00}^2 u_{1,12} u_{2,11} u_{2,30} u_{0,00}^2 \]
\[ + u_{0,13}^3 u_{0,22} u_{1,00}^2 u_{1,20} u_{2,11} u_{2,30}^2 u_{0,00}^2 + 14 u_{0,13}^3 u_{1,00}^2 u_{1,12} u_{2,11} u_{2,30}^2 u_{0,00}^2 \]
\[ - u_{0,13}^2 u_{0,22} u_{1,00}^2 u_{1,12} u_{1,20} u_{2,11} u_{2,30} u_{2,30} u_{0,00}^2 + u_{0,13}^3 u_{1,12} u_{2,11} u_{2,30} u_{0,00}^2 \]
\[ + 3 u_{1,12}^3 u_{0,22} u_{1,20} u_{2,11} u_{2,30} u_{0,00}^2 \]
Example (cont.)

For \( \omega = ((1, -1, 0), (0, 1, -1), (1, -1)) \)

\[
\text{init}_\omega (\text{Res}_A) = u_{0,13}^5 \ u_{1,00}^7 \ u_{2,30}^7
\]

<table>
<thead>
<tr>
<th>( \nu )</th>
<th>( \text{Res}<em>A(\mathbf{f}</em>\nu) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>((1, 1, 2))</td>
<td>( u_{2,30}^6 )</td>
</tr>
<tr>
<td>((-4, -3, 1))</td>
<td>( u_{2,30}^1 )</td>
</tr>
<tr>
<td>((3, -4, 5))</td>
<td>( u_{1,00}^7 )</td>
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<tr>
<td>((8, 2, 7))</td>
<td>( u_{0,13}^5 )</td>
</tr>
<tr>
<td>((2, -3, 4))</td>
<td>( u_{0,13} )</td>
</tr>
<tr>
<td>((-1, 3, 4))</td>
<td>( 1 )</td>
</tr>
</tbody>
</table>
\[ A_0 = \{(0,0),(1,3),(2,2)\}, \quad A_1 = \{(0,0),(1,2),(2,0)\}, \quad A_2 = \{(1,1),(3,0)\} \]
Changing the weight

For $\omega = ((1, 0, 0), (0, 0, 0), (0, 0))$

$$\text{init}_\omega(\text{Res}_A) = u_{1,00}^6 u_{2,30}^4 (u_{1,00}^5 u_{0,13}^3 u_{2,30}^3$$

$$\quad + u_{0,13}^3 u_{0,22}^2 u_{1,20}^2 u_{2,11}^2 u_{2,30} + u_{2,11}^3 u_{0,22}^5 u_{1,12})$$

<table>
<thead>
<tr>
<th>$\mathbf{v}$</th>
<th>$\text{Res}<em>A(\mathbf{f}</em>\mathbf{v})$</th>
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<tbody>
<tr>
<td>(0, 0, 1)</td>
<td>$u_{1,00}^5 u_{0,13}^3 u_{2,30}^3 + u_{0,13}^3 u_{0,22}^2 u_{1,20}^2 u_{2,11}^2 u_{2,30} + u_{2,11}^3 u_{0,22}^5 u_{1,12}$</td>
</tr>
<tr>
<td>(1, 2, 6)</td>
<td>$u_{1,00}^6$</td>
</tr>
<tr>
<td>(0, 1, 2)</td>
<td>$u_{2,30}^4$</td>
</tr>
<tr>
<td>(1, 1, 4)</td>
<td>1</td>
</tr>
</tbody>
</table>
Changing the weight (cont.)

\[A_0=\{(0,0),(1,3),(2,2)\}, \quad A_1=\{(0,0),(1,2),(2,0)\}, \quad A_2=\{(1,1),(3,0)\}\]
Sylvester matrices

Let \( \mathcal{E} \subset \mathbb{Z}^n \) a finite subset and \( \text{RC} \) a row content function on \( \mathcal{E} \): for \( \mathbf{b} \in \mathcal{E} \)

\[
\text{RC}(\mathbf{b}) = (i, a)
\]

with \( 0 \leq i \leq n \) and \( a \in A_i \) such that \( \mathbf{b} - \mathbf{a} + A_i \subset \mathcal{E} \)

For \( \mathbf{b}, \mathbf{b}' \in \mathcal{E} \) set

\[
\mathbf{M}_{\mathbf{b}, \mathbf{b}'} = \text{coefficient of } x^{\mathbf{b}'} \text{ in } x^{\mathbf{b} - \mathbf{a}} f_i
\]

Then

\[\det(\text{Elim}_A) \mid \det(\mathbf{M})\]

Proof. If \( \text{Res}_A(\mathbf{f}) = 0 \), let \( \xi \in (\mathbb{C}^\times)^n \) such that \( f_0 = \cdots = f_n = 0 \). Then

\[
(\xi^{\mathbf{b}})_{\mathbf{b} \in \mathcal{E}} \in \ker(\mathbf{M})
\]

and so \( \det(\mathbf{M}) = 0 \)
Macaulay style formulas

**Multivariate homogeneous resultants.**

- Macaulay (1916)

\[
\text{Res}_{d_0,\ldots,d_n} = \frac{\det(M)}{\det(E)}
\]

with $M$ Sylvester matrix and $E$ block diagonal submatrix

**Sparse eliminants.**

  \[
  \det(M) \text{ a nonzero multiple of } \text{Elim}_A
  \]
  with
  \[
  \deg_{c_0}(\det(M)) = \deg_{c_0}(\text{Res}_A)
  \]

- D’Andrea (2002): Macaulay style formula for $\text{Elim}_A$
A Macaulay style formula for $\text{Res}_A$

We simplify and generalize D’Andrea’s formula to compute $\text{Res}_A$ without imposing the conditions

- $\mathcal{A} = (A_0, \ldots, A_n)$ essential
- $L_A = \mathbb{Z}^n$

Produced by a recursive procedure with input

- $\mathcal{A} = (A_0, \ldots, A_n)$
- $I \subset \{0, \ldots, n\}$ such that $\mathcal{A}_I$ is essential
- $\delta \in \mathbb{Q}^n$ generic

and output a Sylvester matrix $\mathbb{M}$ and a block diagonal submatrix $\mathbb{E}$ of $\mathbb{M}$ such that

$$\text{Res}_A = \frac{\det(\mathbb{M})}{\det(\mathbb{E})}$$
Recall $Q_i = \text{conv}(A_i)$ for $i = 0, \ldots, n$ and set

$$\mathcal{E} := (Q_0 + \cdots + Q_n + \delta) \cap \mathbb{Z}^n$$

The rows and columns of $M$ are indexed by the points in $\mathcal{E}$.
Recursive definition of $RC$, $M$ and $E$

$\mathcal{A} = (A_0, \ldots, A_n)$ with essential subfamily $(A_0, \ldots, A_k)$

$k = 0$  \quad $A_0 = \{a_0\}$ so that $E = (a_0 + Q_1 + \cdots + Q_n + \delta) \cap \mathbb{Z}^n$

Choose generic liftings $\tilde{\omega}_i : A_i \rightarrow \mathbb{R}$

defining polyhedral subdivisions of the $Q_i$'s and of $Q_1 + \cdots + Q_n$

For each cell $C = C_1 + \cdots + C_n$ and $b \in (C + \delta) \cap \mathbb{Z}^n$ set

$$RC(b) = \begin{cases} (i, a) & \text{if } C_i = \{a\} \text{ and } \dim(C_j) > 0 \text{ for } j < i \\ (0, a_0) & \text{otherwise} \end{cases}$$

- $RC$ defines a Sylvester matrix $M$
- $E$ given by $\{b \in E \mid RC(b) = (i, a) \text{ with } i \neq 0\}$

In this case $Res_\mathcal{A} = c_0^{MV(Q_1, \ldots, Q_n)} = \frac{\det(M)}{\det(E)}$
Choose $a_0 \in A_0$ and $\omega = (\omega_0, \omega_1, \ldots, \omega_n)$ given by

- $\omega_0(a_0) = 0$ and $\omega_0(a) = 1$ for $a \in A_0$, $a \neq a_0$
- $\omega_i(a) = 1$ for $a \in A_i$ and $i = 1, \ldots, n$

Let $v_0, \ldots, v_N \in \mathbb{Z}^{n+1}$ primitive inner normals to the facets of the lower envelope of $Q_0, \omega_0 + \cdots + Q_n, \omega_n$. Then

- if $v_0 = (0, 1)$, $A_0, v_0 = \{a_0\}$ is an essential subfamily of $(A_0, v_0, \ldots, A_n, v_0)$
- if $v_j \neq (0, 1)$, there is an essential subfamily contained in $(A_1, v_j, \ldots, A_k, v_j)$

For $b \in E$ in the cell associated to $v_j$, define $RC(b)$ from the function $RC$ associated to $(A_0, v_j, \ldots, A_n, v_j)$ and this essential subfamily
RC defines a Sylvester matrix $\mathbf{M}$

For $j = 0, \ldots, N$, let $\mathbf{M}_{\mathbf{v}_j}$ be the matrix associated to $(\mathcal{A}_{0,\mathbf{v}_j}, \ldots, \mathcal{A}_{n,\mathbf{v}_j})$ and its marked essential subfamily, and $\mathbf{E}_{\mathbf{v}_j}$ its corresponding submatrix, indexed by some points of $\mathcal{E}$.

Set $\mathbf{E}$ as the submatrix of $\mathbf{M}$ with rows and columns are indexed by the points in $\mathcal{E}$ which index the $\mathbf{E}_{\mathbf{v}_j}$'s.

**Theorem**

$$\text{Res}_\mathcal{A} = \frac{\det(\mathbf{M})}{\det(\mathbf{E})}$$
Example

\[ A_0 = \{(0,0),(1,3),(2,2)\}, \quad A_1 = \{(0,0),(1,2),(2,0)\}, \quad A_2 = \{(1,1),(3,0)\} \]

\[ f_0 = a_0 + a_1 xy^3 + a_2 x^2 y^2, \quad f_1 = b_0 + b_1 xy^2 + b_2 x^2, \quad f_2 = c_0 xy + c_1 x^3 \]
\[
\det(M) = b_1 c_0^3 \cdot \left( b_0^7 a_1^5 c_1^7 + a_1^3 a_0^2 b_2^7 c_0^6 c_1 - 2 c_1^2 c_0^5 a_2^4 b_0^3 a_0 b_1 b_2^3 + c_0^7 a_2^3 a_1^2 b_1 b_2^6 - c_1^2 c_0^5 a_1^2 b_0^2 b_1 b_2^5 a_2 \\
- 2 b_0^3 a_1^2 a_0 b_2^4 c_0^4 c_1^3 + b_0^6 a_1^3 a_2 b_2 c_0^2 c_1^5 + 2 b_0^4 a_1^2 a_2 a_0 b_1 b_2^2 c_0^4 c_1^4 - b_0^2 a_1 a_2^2 b_1^2 b_2^3 c_0^4 c_1^3 \\
+ 14 b_0^3 a_2^2 b_1^2 c_0^2 c_1^5 a_1^3 b_2^2 - 5 b_0^5 a_0 b_1^2 c_0^2 c_1^5 a_2^3 a_1 + 6 b_0^3 a_2^2 b_1^2 c_0^3 c_1^4 a_2 b_2 - 7 b_0^3 a_0^3 b_1^3 c_0^3 c_1^4 a_2^2 b_2^3 \\
+ 5 b_0^4 a_0^2 b_1^3 c_0^6 a_2 a_1^2 - 13 b_0^2 a_0 b_1^3 c_0^5 c_1^2 b_2 a_2 + 3 a_0^3 b_1^3 c_0^5 c_1^2 a_2^2 b_2^4 - 7 b_0^5 a_0 b_1 c_0^6 a_2 b_2 \\
+ 3 a_0^4 b_1^5 c_0^3 c_1^4 b_2 a_2 + c_0^4 c_1^3 a_2^3 b_0^6 b_1 + a_0^5 b_1^7 c_0^6 \right)
\]
Proof (sketch)

Proposition

- $\text{init}_\omega(\det(M)) = \prod_{j=0}^{N} \det(M_{v_j})$.
- $\det(E) = \prod_{j=0}^{N} \det(E_{v_j})$.
- $\det(M) = P \cdot \text{Res}_A$ with $P \in \mathbb{Z}[c_1, \ldots, c_n]$.

Hence

\[
\frac{\det(M)}{\text{Res}_A} = \frac{\text{init}_\omega(\det(M))}{\text{init}_\omega(\text{Res}_A)} = \frac{\prod_{j=0}^{N} \det(M_{v_j})}{\prod_{j=0}^{N} \text{Res}_A(v_j)} = \prod_{j=0}^{N} \det(E_{v_j}) = \det(E)
\]
Thanks!
and hope to see you soon again...