The tropical analogue of the Helton–Nie conjecture is true

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Outline

- Section I: Spectrahedra and the Helton–Nie conjecture
- Section II: Tropicalization of convex semialgebraic sets
- Section III: Tropical Helton–Nie conjecture

Section I: Spectrahedra
and the Helton–Nie conjecture

Spectrahedra

Definition (spectrahedron)

Given symmetric matrices $Q^{(0)}, \dots, Q^{(n)} \in \mathbb{R}^{m \times m}$, the associated **spectrahedron** is defined as

$$\mathcal{S} = \{x \in \mathbb{R}^n \colon Q^{(0)} + x_1 Q^{(1)} + \dots + x_n Q^{(n)} \text{ is positive semidefinite}\} .$$

Example: an *elliptope* in $\mathbb{R}^3$ is a spectrahedron defined by

$$Q(x) = \begin{bmatrix} 1 & x_1 & x_2 \\ x_1 & 1 & x_3 \\ x_2 & x_3 & 1 \end{bmatrix} .$$

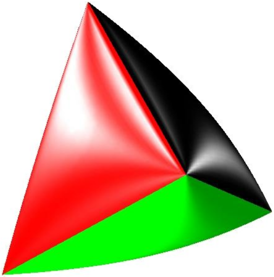

Figure: 3D elliptope. Source: http://www.math.uni-frankfurt.de/~rostalsk/pmwiki

Semidefinite programming (SDP)

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- Applications of SDP: combinatorial optimization, nonconvex optimization, operations research, control theory...
- Approximate solutions to well-structured SDP problems can be found in polynomial time by ellipsoid and interior-point methods.

The Helton–Nie conjecture in a nutshell

Nemirovski's question (2006)

Which convex semialgebraic sets arise as projections of spectrahedra?

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Theorem (Scheiderer, Dec 2016)

The Helton–Nie conjecture is false. Projected spectrahedra have some nontrivial properties.[a]

[a] C. Scheiderer. *Semidefinitely representable convex sets.* arXiv:1612.07048. 2016.

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For any convex semialgebraic set over the Puiseux series there exists a projected spectrahedron that has the same image by valuation.

- To do that, we use a technique from game theory (Zwick–Paterson lemma).

Section II: Tropicalization of convex semialgebraic cones

- A (formal generalized real) **Puiseux series** is a series of form

$$\boldsymbol{x} = \boldsymbol{x}(t) = \sum_{i=1}^{\infty} c_i t^{\alpha_i} \, ,$$

where the sequence $(\alpha_i)_i \subset \mathbb{R}$ is strictly decreasing and either finite or unbounded and c_i are real.

[1] L. van den Dries and P. Speissegger. "The real field with convergent generalized power series". In: *Trans. Amer. Math. Soc.* 350.11 (1998), pp. 4377–4421.

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- The subset of absolutely converging (for t large enough) Puiseux series forms a real closed field,[1] denoted here by $\mathbb{K}$.

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- The subset of absolutely converging (for t large enough) Puiseux series forms a real closed field,[1] denoted here by $\mathbb{K}$.
- We say that $\boldsymbol{x} \geq \boldsymbol{y}$ if $\boldsymbol{x}(t) \geq \boldsymbol{y}(t)$ for all t large enough. This is a linear order on $\mathbb{K}$.

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Valuation

- With every point $\boldsymbol{x} = \boldsymbol{x}(t) = \sum_{i=1}^{\infty} c_i t^{\alpha_i} \in \mathbb{K}$ we associate its **valuation** defined as the highest exponent occurring in $\boldsymbol{x}$,

$$\mathrm{val}(\boldsymbol{x}) = \lim_{t\to\infty} \log_t |\boldsymbol{x}(t)| = \alpha_1 \quad (\text{and } \mathrm{val}(0) = -\infty)\,.$$

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- If $\boldsymbol{\mathcal{S}} \subset \mathbb{K}^n$ is semialgebraic, then $\operatorname{val}(\boldsymbol{\mathcal{S}}) \subset (\mathbb{R} \cup \{-\infty\})^n$ has polyhedral structure.
- This follows from general results of model theory.[2,3]

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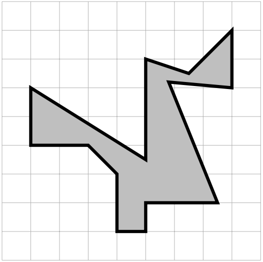

Figure: Tropical semialgebraic set.

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Tropicalization of cones and min-max operators

- Suppose that $\boldsymbol{\mathcal{S}} \subset \mathbb{K}^n_{\geq 0}$ is a convex semialgebraic cone (*cone* means that $\boldsymbol{\lambda x} + \boldsymbol{\mu y} \in \boldsymbol{\mathcal{S}}$ for all $\boldsymbol{\lambda}, \boldsymbol{\mu} \geq 0$).

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Theorem

The set $\mathcal{S} \subset \mathbb{R}^n$ arises as a tropicalization of a convex semialgebraic cone if and only if we have $\mathcal{S} = \{x \in \mathbb{R}^n : x \leq F(x)\}$, where $F \colon \mathbb{R}^n \to \mathbb{R}^n$ is an operator of the form

$$\forall k, \ (F(x))_k = \min_{i \in [M_k]} \max_{u \in U_{ki}} \left(A_k^{(u)} x + b_k^{(u)}\right),$$

where $A^{(1)}, \dots, A^{(p)} \in \mathbb{Q}^{n \times n}$ is a sequence of stochastic matrices, $b^{(u)} \in \mathbb{R}^n$ for all $u \in [n]$, $M_k \geq 1$ for all $k \in [n]$, and U_{ki} is a subset of $[p]$ for every $k \in [n]$ and $i \in [M_k]$.

Example

Take $n = 3$, $p = 2$, $M_k = 1$, $U_k = \{1, 2\}$ for all $k \in \{1, 2, 3\}$,

$$A^{(1)} = \begin{bmatrix} 0 & 0 & 1 \\ 1/4 & 0 & 3/4 \\ 1 & 0 & 0 \end{bmatrix} \quad b^{(1)} = \begin{bmatrix} 1 \\ 3/4 \\ 0 \end{bmatrix},$$

$$A^{(2)} = \begin{bmatrix} 0 & 1/3 & 2/3 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad b^{(2)} = \begin{bmatrix} 4/3 \\ 2\pi \\ 0 \end{bmatrix}.$$

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Then $\mathcal{S}$ is the set of all points $x \in \mathbb{R}^3$ verifying

$$x_1 \leq \max\left\{x_3 + 1, \frac{1}{3}x_2 + \frac{2}{3}x_3 + \frac{4}{3}\right\},$$

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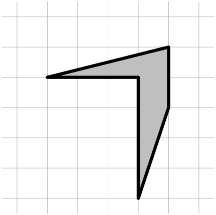
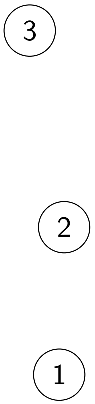


Figure: Tropicalization of a cone (for $x_3 = 0$).

Min-max operators and directed graphs

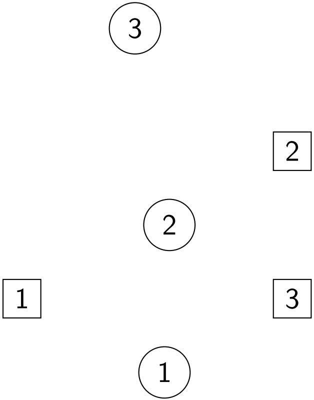
Description of a graph

The min-max operators can be described using directed graphs.



Description of a graph

Our graph $\vec{\mathcal{G}} = (V, E)$ has three types of vertices: Min vertices V_{Min},



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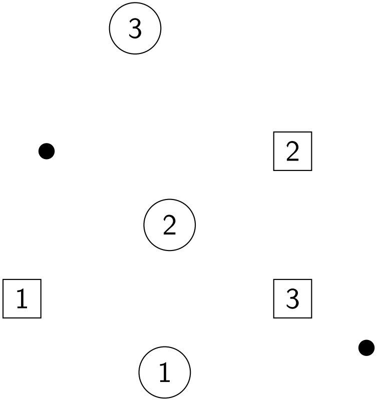

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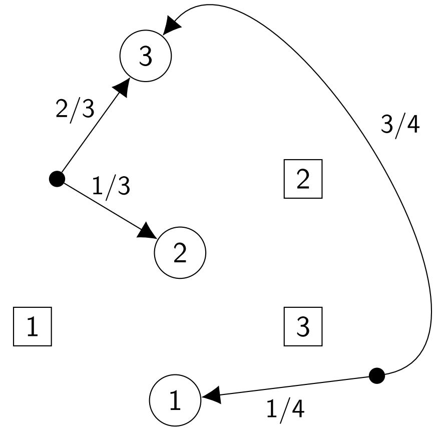
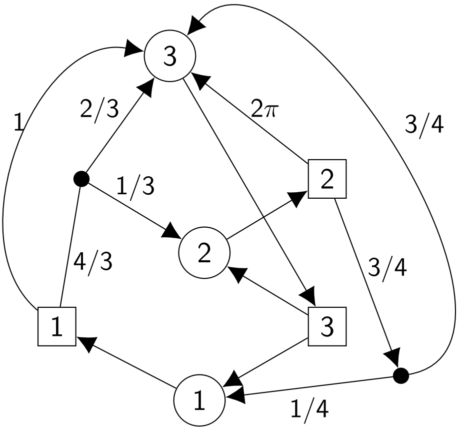


Description of a graph

Every Random vertex $v \in V_{\mathrm{Rand}}$ is equipped with a rational probability distribution on the set of its outgoing edges $E(v)$. (This corresponds to stochastic matrices $A^{(u)}$).

Min-max operators and directed graphs



Description of a graph

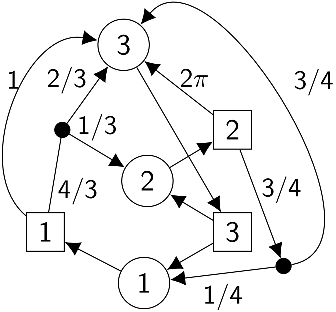
Every outgoing edge of a Max (resp. Min) vertex is equipped with a real number r_e. (This corresponds to vectors $b^{(u)}$).

Min-max operators and directed graphs

The associated operator $F \colon \mathbb{R}^{V_{\mathrm{Min}}} \to \mathbb{R}^{V_{\mathrm{Min}}}$ is defined as

$$(F(x))_v := \min_{e \in E(v)} \Big(r_e + \sum_{w \in V_{\mathrm{Max}}} p_w^e \max_{e' \in E(w)} \big(r_{e'} + \sum_{u \in V_{\mathrm{Min}}} p_u^{e'} x_u \big) \Big) \, ,$$

where p_v^e denotes the probability that a token moving on the graph according to the probability distributions given by Random vertices goes to v starting from the head of e.



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Here, we obtain the familiar operator

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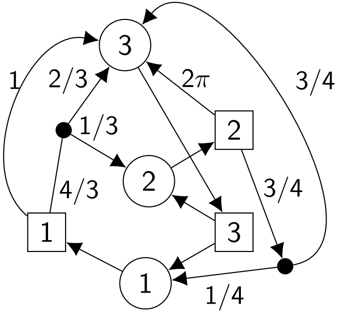

Tropical Metzler spectrahedra

Proposition (Tropical Metzler spectrahedra)

Suppose that the graph $\vec{\mathcal{G}}$ has the following property: each Random vertex has exactly two outgoing edges with probability distribution $(1/2, 1/2)$ and these edges have Max vertices as their heads. Then, the set $\{x \colon x \leq F(x)\}$ is a tropicalization of a spectrahedral cone.

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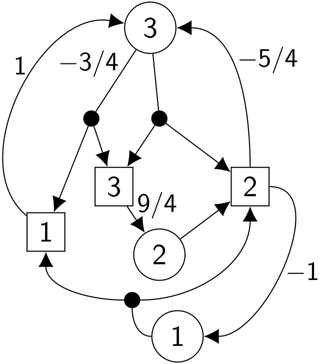

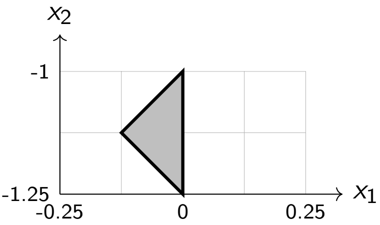

Figure: Tropical Metzler spectrahedral cone (for $x_3 = 0$).

Section III: Tropical Helton–Nie conjecture

Idea of the proof

- The idea of proving the tropical Helton–Nie conjecture reduces to the following:

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- The idea of proving the tropical Helton–Nie conjecture reduces to the following:
- We want to go from an arbitrary graph $\vec{\mathcal{G}}$ to a graph which has only probabilities $(1/2, 1/2)$ at Random vertices, keeping control over the tropical cone defined by the graph.

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- We want to go from an arbitrary graph $\vec{\mathcal{G}}$ to a graph which has only probabilities $(1/2, 1/2)$ at Random vertices, keeping control over the tropical cone defined by the graph.
- It turns out that this can be done efficiently (in polynomial time) using the Zwick–Paterson lemma from the theory of stochastic games.[4]

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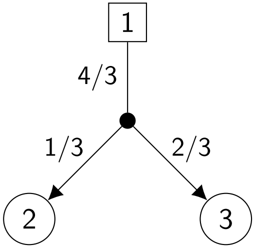
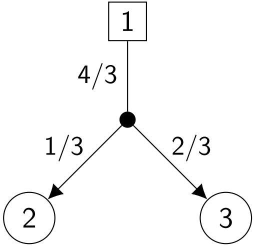
Zwick–Paterson lemma in action

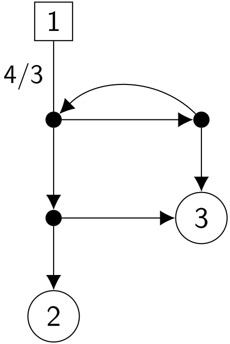


Zwick–Paterson lemma

We want to simulate the probability distribution $(1/3, 2/3)$ using only nodes with distribution $(1/2, 1/2)$.

Zwick–Paterson lemma in action

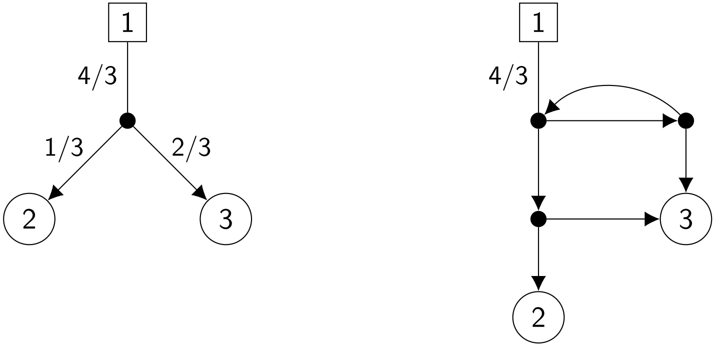




Zwick–Paterson lemma

Take the new graph depicted above. All probabilities are equal to $(1/2, 1/2)$.

Zwick–Paterson lemma in action

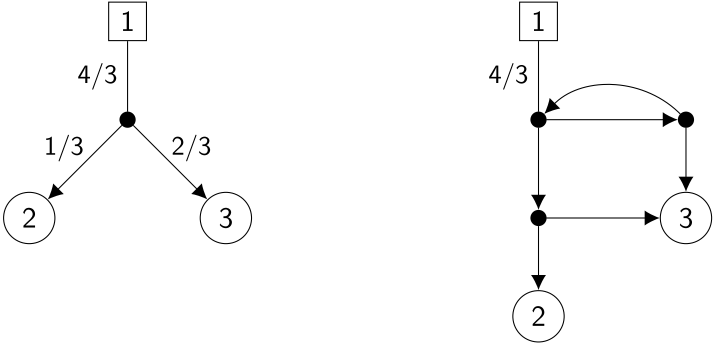


Zwick–Paterson lemma

The probability of reaching (2) is equal to

$$\frac{1}{4} + \frac{1}{4} \cdot \frac{1}{4} + \left(\frac{1}{4}\right)^2 \cdot \frac{1}{4} + \cdots = \frac{1}{4} \cdot \frac{1}{1 - \frac{1}{4}} = \frac{1}{3} \, .$$

Zwick–Paterson lemma in action



Zwick–Paterson lemma

This can be done for arbitrary rational probabilities (and we add only polynomially many vertices to the graph).

Constructing lifts to tropical spectrahedra

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- We show that in order to construct a lift into a tropical spectrahedral cone, it is enough to add vertices on the edges of this graph (so that every edge outgoing from a Random vertex has a Max vertex as its head).

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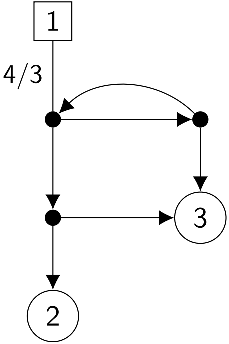

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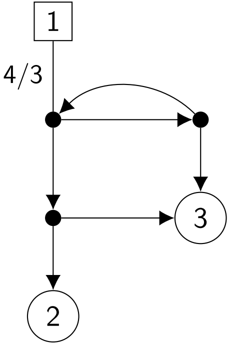

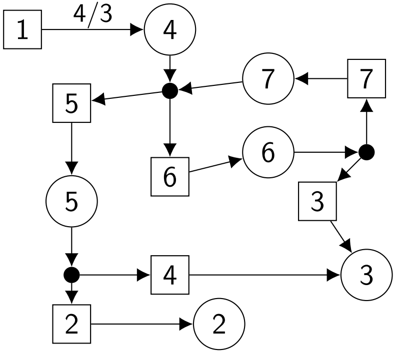

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- We show that the analogue of the Helton–Nie conjecture is true in the tropical geometry.
- In other words, the valuation function is not able to differentiate projected spectrahedra from general convex semialgebraic sets.
- Is there a more precise notion of tropicalization that captures this difference?
- Is there a tighter connection between tropical and classical Helton–Nie conjecture? (E.g., some tropical bounds for the dimension of the smallest lift?)

Thank you for your attention

- X. Allamigeon, S. Gaubert, and M. Skomra. "Solving generic nonarchimedean semidefinite programs using stochastic game algorithms". In: *Proceedings of ISSAC 2016*. ACM, 2016, pp. 31–38
- X. Allamigeon, S. Gaubert, and M. Skomra. *Tropical spectrahedra*. arXiv:1610.06746. 2016
- X. Allamigeon, S. Gaubert, and M. Skomra. *The tropical analogue of the Helton–Nie conjecture is true*. Presented at MEGA 2017

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A convex set is a projected spectrahedron if and only if its homogenization is a projected spectrahedron. Moreover, a convex hull of finitely many projected spectrahedra is a projected spectrahedron.[a]

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- The proof is so elementary that is works without any change in the tropical geometry.
- It allows to prove the general case of the tropical Helton–Nie conjecture.

References I

- D. Alessandrini. "Logarithmic limit sets of real semi-algebraic sets". In: *Adv. Geom.* 13.1 (2013), pp. 155–190.
- X. Allamigeon, S. Gaubert, and M. Skomra. "Solving generic nonarchimedean semidefinite programs using stochastic game algorithms". In: *Proceedings of ISSAC 2016*. ACM, 2016, pp. 31–38.
- X. Allamigeon, S. Gaubert, and M. Skomra. *The tropical analogue of the Helton–Nie conjecture is true*. Presented at MEGA 2017.
- X. Allamigeon, S. Gaubert, and M. Skomra. *Tropical spectrahedra*. arXiv:1610.06746. 2016.
- L. van den Dries and P. Speissegger. "The real field with convergent generalized power series". In: *Trans. Amer. Math. Soc.* 350.11 (1998), pp. 4377–4421.

T. Netzer and R. Sinn. *A note on the convex hull of finitely many projections of spectrahedra*. arXiv:0908.3386. 2009.

J. Pas. "Uniform *p*-adic cell decomposition and local zeta functions". In: *J. Reine Angew. Math.* 399 (1989), pp. 137–172.

C. Scheiderer. *Semidefinitely representable convex sets*. arXiv:1612.07048. 2016.

U. Zwick and M. Paterson. "The complexity of mean payoff games on graphs". In: *Theoret. Comput. Sci.* 158.1–2 (1996), pp. 343–359.