PRIMARY DECOMPOSITION OF CERTAIN DETERMINANTAL IDEALS

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ABSTRACT

Our aim is to compute primary decomposition of ideals of the form $I_1(XY)$. We use the technique of Gröbner bases. This work forms the main portions of the articles [3], [4].

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INTRODUCTION

Let K be a field. Let $\{x_{ij};\ 1 \leq i \leq m,\ 1 \leq j \leq n\}$, $\{y_j;\ 1 \leq j \leq n\}$ be indeterminates over K and $R = K[x_{ij}, y_j]$. Let X denote an $m \times n$ matrix with entries in the ideal $\langle \{x_{ij};\ 1 \leq i \leq m,\ 1 \leq j \leq n\}\rangle$. Let $Y = (y_j)_{n\times 1}$ be the generic $n \times 1$ column matrix. Let $I_1(XY) = \langle g_1, \ldots, g_m\rangle$, denote the ideal generated the entries of the $m \times 1$ matrix XY. Ideals of the form $I_1(XY)$ were studies by J. Herzog [2] in 1974, which motivated our work.

MAIN THEOREMS

Suppose that $m = n$ and $X = (x_{ij})$ is generic or generic symmetric. Define the ideals $\mathcal{I}_t = \langle g_1, \ldots, g_t\rangle$, with $1 \leq t \leq m-1$ and $\mathcal{I} := \langle g_1, \ldots, g_m\rangle$. Similarly, if $m = n+1$ and $X = (x_{ij})$ is generic we define $\mathcal{J} := \langle g_1, \ldots, g_m\rangle$.

Theorem 1 [Primality.]

(1) Let $m = n$ and $X = (x_{ij})$ be generic or generic symmetric, then $\mathcal{I}_t = \langle g_1, \ldots, g_t\rangle$,with $1 \leq t \leq m-1$, is a prime ideal in R. Moreover, $g_1, \ldots, g_m$ is a regular sequence in R.

(2) If $m < n$ and $X = (x_{ij})$ is generic, then $\mathcal{I} = \langle g_1, \ldots, g_m\rangle$ is a prime ideal.

(3) If $m = n$ and X is generic or generic symmetric then $\mathcal{I}$ is not a prime ideal.

(4) If $m = n+1$ and X is generic then $\mathcal{J}$ is not a prime ideal.

Theorem 2 [Primary Decomposition.]

(1) The primary decomposition for the ideal $\mathcal{I}$ is given by $\mathcal{I} = \langle y_1, \cdots, y_n\rangle \cap \langle g_1, \cdots, g_n, \Delta\rangle$, where Δ denotes the determinant of X.

(2) The primary decomposition of the ideal $\mathcal{J}$ is given by $\mathcal{J} = \langle y_1, \cdots, y_n\rangle \cap \langle g_1, \cdots, g_n, \Delta_1, \cdots, \Delta_{n+1}\rangle$, where Δ_i denotes the determinant of the $n \times n$ matrix formed by removing the i-th row of the matrix $\widetilde{X}$.

GRÖBNER BASES

(i) $C_k := \{\mathbf{a} = (a_1, \cdots, a_k) \mid 1 \leq a_1 < \cdots < a_k \leq m\}$; denotes the collection of all ordered k-tuples from $\{1, \cdots, n\}$.

(ii) Given $\mathbf{a} = (a_1, \ldots, a_k) \in C_k$;

- $X^{\mathbf{a}} = [a_1, \cdots, a_k | 1, 2, \ldots, k]$ denotes the $k \times k$ minor of the matrix X, with $a_1, \ldots, a_k$ as rows and $1, \ldots, k$ as columns.
- $S_k := \{X^{\mathbf{a}} : \mathbf{a} \in C_k\}$ and I_k denotes the ideal generated by S_k in the polynomial ring R;
- $X^{\mathbf{a},m} := [a_1, \cdots, a_k | 1, \cdots, k-1, m]$; if $m \geq k$;
- $\widetilde{X^{\mathbf{a}}} = \sum_{m\geq k}[a_1, \cdots, a_k | 1, \cdots, k-1, m]y_m = \sum_{m\geq k} X^{\mathbf{a},m} y_m$;
- $\widetilde{S}_k := \{\widetilde{X^{\mathbf{a}}} : X^{\mathbf{a}} \in S_k\}$ and $\widetilde{I}_k$ denotes the ideal generated by $\widetilde{S}_k$ in the polynomial ring R;
- $G_k = \cup_{i\geq k}\widetilde{S}_i$;
- $G = \cup_{k\geq 1} G_k$
- $X^{\mathbf{a}}_r := [a_1, a_2, \cdots, \hat{a_r}, a_{r+1} \cdots, a_k | 1, 2, \cdots, k-1]$, if $k \geq 2$.

Theorem 3. The set G_k is a reduced Gröbner Basis for the ideal $\widetilde{I}_k$, with respect to the monomial order $y_1 > y_2 > \cdots > y_n > x_{ij}$ for all i, j, such that $x_{ij} > x_{i'j'}$ if $i < i'$ or if $i = i'$ and $j < j'$. In particular, $\mathcal{G} = G_1$ is a reduced Gröbner Basis for the ideal $\widetilde{I}_1 = \mathcal{I}$ (respectively $\mathcal{J}$).

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COMPLETE IRREDUCIBILITY

Let P be a commutative ring with identity. Let $\mathfrak{a}$ be a prime ideal of P. Let $\Gamma_{\mathfrak{a}} := \{f \in P[x] \mid \delta f \neq 0,\ a = \mathrm{lc}(f) \notin \mathfrak{a}\}$, where δf denotes the degree of f and lc denotes the leading coefficient of f, with respect to the indeterminate x. Given $f \in P[x]$, let

$$[\mathfrak{a}, f] := \{g \in P[X] \mid g\langle a^e\rangle \subset \mathfrak{a}[X] + \langle f\rangle \text{for some integer} \quad e \geq 0.\}.$$

A polynomial $f \in \Gamma_{\mathfrak{a}}$ is $\Gamma_{\mathfrak{a}}$ *completely irreducible* if the following criteria holds: If there exist $b \in P$, $g \in \Gamma_{\mathfrak{a}}$, $h \in P[X]$, such that $fb \notin P[X]$ and $fb - gh \in P[X]$ then $\delta g = \delta f$. Let $Q = P[x_1, \ldots, x_n]$. For $i = 1, \ldots, n$, let $f_i \in R[x_1, \ldots, x_{i-1}][x_i]$, with $a_i = \mathrm{lc}(f_i) \in P[x_1, x_2, \cdots, x_{i-1}]$, with respect to the indeterminate x_i. Then, $[\mathfrak{a}, f_1, \cdots, f_n]$ is a prime ideal. If $\mathfrak{a} = \langle 0\rangle$ is prime then $[\langle 0\rangle, f_1, \cdots, f_n]$ is written as $[f_1, \cdots, f_n]$. The sequence $(f_1, f_2, \cdots, f_n)$ defined above is said to be *completely irreducible (mod $\mathfrak{a}$)* if f_1 is $\Gamma_{\mathfrak{a}}$ completely irreducible and f_{i+1} is $\Gamma_{\mathfrak{a}_i}$ completely irreducible as a polynomial in x_{i+1}, where $\mathfrak{a}_0 = \mathfrak{a}$ and $\mathfrak{a}_i = [\mathfrak{a}_{i-1}, f_i]$, for every $0 \leq i \leq n-1$.

Theorem 4. [2.5 in [1]] Let $\mathfrak{a}$ be a prime ideal of P and $(f_1, \ldots, f_n)$ a completely irreducible sequence (mod $\mathfrak{a}$). Then $\mathfrak{q} = [\mathfrak{a}, f_1, \ldots, f_n]$ is a prime ideal of S such that $\mathfrak{q} \cap P = \mathfrak{a}$.

PRIMALITY OF $\mathcal{I}_t$

Lemma 1. Let $m = n$ and $X = (x_{ij})$ be generic or generic symmetric.

(1) The sequence $(g_1, \ldots, g_m)$ is completely irreducible (mod $\langle 0\rangle$) and the ideal $[g_1, \cdots, g_m]$ is a prime ideal.

(2) The sequence $(g_1, \ldots, g_t)$ is completely irreducible (mod $\langle 0\rangle$) and the ideal $[g_1, \cdots, g_t]$ is a prime ideal, for every $t = 1, \ldots, m$.

Proof. The proof follows from the definition of complete irreducibility and Theorem 4. □

Theorem 5. Let $m = n$ and $X = (x_{ij})$ be generic or generic symmetric. Then, $[g_1, \cdots, g_t] = \langle g_1, \cdots, g_t\rangle$, for every $t = 1, \ldots, m-1$.

PRIMARY DECOMPOSITIONS OF $\mathcal{I}$ AND $\mathcal{J}$

Suppose that X generic with $m = n$ or $m = n+1$. It is easy to prove that both $\mathcal{I}$ and $\mathcal{J}$ are not primes. We now show the main steps towards constructing the primary decompositions of the ideals $\mathcal{I}$ and $\mathcal{J}$ using their Gröbner basis and the results of complete irreducibility developed in [1].

Theorem 6. Let $I = \langle g_1, \ldots, g_n, \Delta\rangle$ and $\mathfrak{G} = (\mathcal{G} \setminus G_n) \cup \{\Delta\}$. Then $\mathfrak{G}$ is a Gröbner basis for I, with respect to the lexicographic monomial order given by $y_1 > \cdots > y_n > x_{11} > x_{12} > \cdots x_{n,(n-1)} > x_{n,n}$ on R.

Lemma 2. Suppose that $gy_i \in \langle g_1, \cdots, g_n, \Delta\rangle$, then $g \in \langle g_1, \cdots, g_n, \Delta\rangle$.

Lemma 3. $\Delta y_i = \sum_{j=1}^{n} A_{ji} g_j$, where A_{ji} is the cofactor of x_{ji} in X.

Theorem 7. $\langle g_1, \cdots, g_n, \Delta\rangle$ is a prime ideal.

Lemma 4. The minimal prime ideals containing $\mathcal{I}$ are $\langle y_1, \cdots, y_n\rangle$ and $\langle g_1, \cdots, g_n, \Delta\rangle$.

The above theorem tells us that $\sqrt{\mathcal{I}} = \langle y_1, \cdots, y_n\rangle \cap \langle g_1, \cdots, g_n, \Delta\rangle$. We now show that $\sqrt{\mathcal{I}} = \mathcal{I}$ in the following Lemma.

Lemma 5. $\sqrt{\mathcal{I}} = \mathcal{I}$.

The proofs of the theorem for the ideal $\mathcal{I}$ follows from Lemmas 4 and 5. The proof for $\mathcal{J}$ would be similar.

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