Algorithms for Tight Spans and Tropical Linear Spaces

Benjamin Schröter

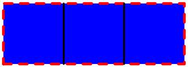
TU Berlin

Nice, June 14, 2017

joint w/ Michael Joswig
and Simon Hampe

Tight Spans and Generalizations

- they carry the relevant information of a subdivision
- are important in pyhlogenetics and for metric spaces
- as well as in polyhedral and tropical geometry

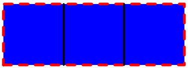


A subdivision

the tight span

Tight Spans and Generalizations

- they carry the relevant information of a subdivision
- are important in pyhlogenetics and for metric spaces
- as well as in polyhedral and tropical geometry



A subdivision

the tight span

A new effective algorithm via

- construction/modification of closure operators
- usage of an existing algorithm

Generalizations and new examples for
Speyer's f-vector conjecture and homology of tropical linear spaces.

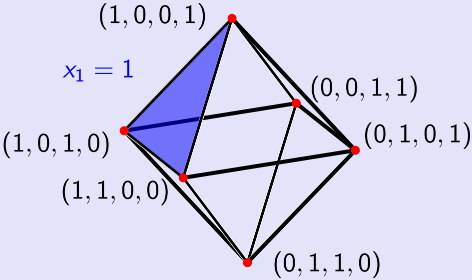
Polytopes and their Faces

A *polytope* is the convex hull of finitely many points $v_1, \dots v_k \in \mathbb{R}^n$.

$$P = \left\{ x \in \mathbb{R}^n \;\middle|\; x = \sum \lambda_i v_i, \text{ with } \sum \lambda_i = 1 \text{ and } \lambda_i \geq 0 \right\}$$

A *face* of the polytope is the convex hull of the points that satisfy a given affine equation $\sum \alpha_j x_j = \beta$.

Example



The Octahedron has 28 faces:

The empty set, 6 vertices, 12 edges, 8 triangles and the octahedron itself.

Polyhedral Complexes, Subdivisions and their Cells

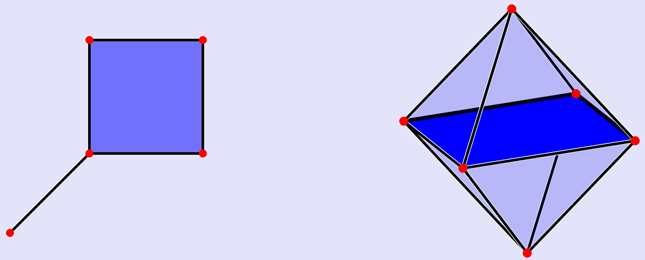
A finite collections of polytopes/polyhedra is a *polyhedral complex* if

- the faces of a cell are in the collection,
- the intersection of two polytopes is a face for both.

It is a *polyhedral subdivision* if additionally

- the union of all cells is a polytope.

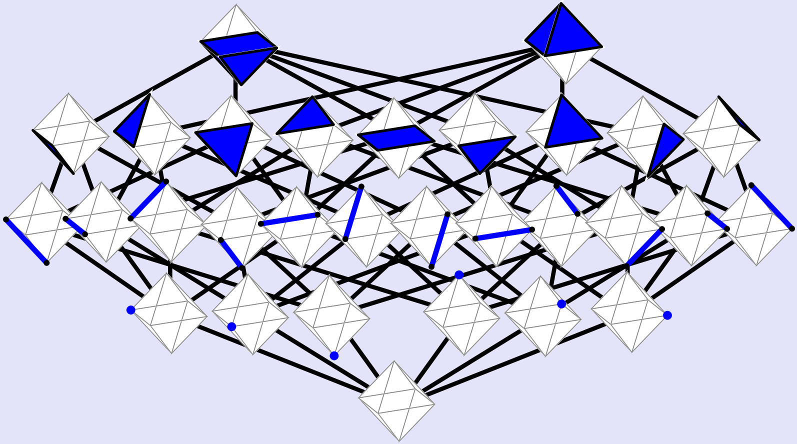
Example



Their Face Lattices

Both the faces of a polytope and the cells in a subdivision form a partially ordered sets w.r.t. inclusion and *meet-semilattice* w.r.t. intersection.

Example



A General Framework: Closure Systems

A *closure operator* on the set S is a function $\mathrm{cl} : \mathcal{P}(S) \to \mathcal{P}(S)$, such that

- $A \subseteq \mathrm{cl}(A)$,
- $A \subseteq B$ implies $\mathrm{cl}(A) \subseteq \mathrm{cl}(B)$ and
- $\mathrm{cl}(\,\mathrm{cl}(A)\,) = \mathrm{cl}(A)$.

A set A is *closed* if $A = \mathrm{cl}(A)$.

Example

- Faces of a polytope are closed.
- Cells of a subdivision are closed.
- In a topological space complements of open sets are closed.
- ...

Duality of Polytopes and Subdivisions

The *dual of a polytope P*, with interior point 0 is the polytope:

$$Q = \left\{ y \in \mathbb{R}^n \;\middle|\; \sum x_i\, y_i \leq 1 \text{ for all } x \in P \right\}.$$

There is an inclusion reversing bijection between the faces of P and Q.

Duality of Polytopes and Subdivisions

The *dual of a polytope P*, with interior point 0 is the polytope:

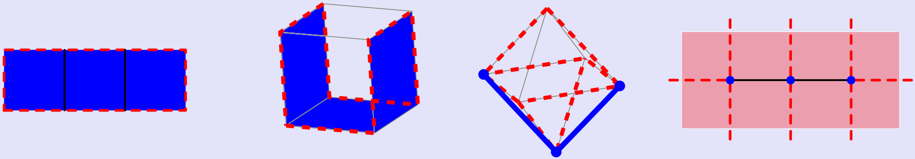
$$Q = \left\{ y \in \mathbb{R}^n \;\middle|\; \sum x_i\, y_i \leq 1 \text{ for all } x \in P \right\}.$$

There is an inclusion reversing bijection between the faces of P and Q.

If Σ is a collection of faces of a polytope, then the *dual of* Σ is a complex, whose cells are the dual faces of those in Σ.

The boundary of the complex Σ is mapped to unbounded polyhedra.

Example



Tight Spans and Their Closure Operators

The dual of a subdivision Σ is an abstract complex on the set S_Σ of maximal cells unified with the maximal cells in the boundary.
The dual closure operator on S_Σ is

$$\mathrm{cl}^{\Sigma}(A) = \begin{cases} \emptyset & \text{if } A = \emptyset \ , \\ \{B \in S_\Sigma \mid \bigcap_{\sigma \in A} \sigma \subseteq B\} & \text{otherwise} \ . \end{cases}$$

Tight Spans and Their Closure Operators

The dual of a subdivision Σ is an abstract complex on the set S_Σ of maximal cells unified with the maximal cells in the boundary.
The dual closure operator on S_Σ is

$$\mathrm{cl}^\Sigma(A) = \begin{cases} \emptyset & \text{if } A = \emptyset \ , \\ \{B \in S_\Sigma \mid \bigcap_{\sigma \in A} \sigma \subseteq B\} & \text{otherwise} \ . \end{cases}$$

The *tight span* is the subcomplex restricted to the maximal cells, i.e., the bounded cells in the dual complex.

$\mathrm{cl}^\Sigma_\Delta(A) = S_\Sigma$ if and only if A is contained in the boundary.

Matroid Subdivisions and Tropical Linear Spaces

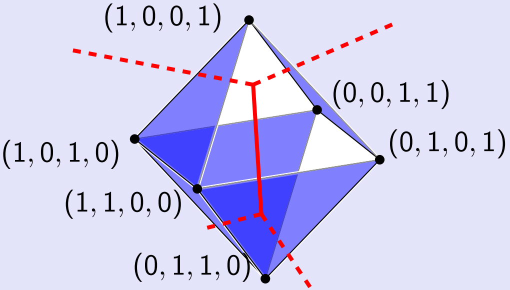
A subpolytope of the hypersimplex

$$\Delta(d,n) = \left\{ x \in [0,1]^n \;\middle|\; \sum x_i = d \right\}$$

without any new edges is called *matroid polytope*.

Cells in a matroid subdivision that are not contained in a coordinate hyperplane form the *tropical linear space*.

Example



Extended Tight Spans and Tropical Linear Spaces

If Γ is a collection of maximal cells in the boundary, then

$$\mathrm{cl}_{\Gamma}^{\Sigma} = \begin{cases} \emptyset & \text{if } A = \emptyset \ , \\ S_{\Sigma} & \text{if } A \subseteq \sigma \in \Gamma \ , \\ \{B \in S_{\Sigma} \mid \bigcap_{\sigma \in A} \sigma \subseteq B\} & \text{otherwise} \ . \end{cases}$$

defines a closure operator on the set S_{Σ}. We call the resulting complex the *extended tight span*.

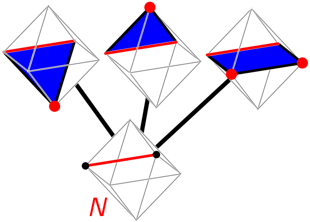
Choosing Γ as the maximal boundary cells in a coordinate hyperplane gives the tropical linear space.

For experts:

- Equipped with the coarsest polyhedral structure.
- Non-trivial valuation is covered.
- Non-regular tropical linear spaces are included.

Ganter's Algorithm

procedure HASSE(Set S, ClosureOperator cl)
    $H \leftarrow$ empty graph
    Queue $\leftarrow$ [cl($\emptyset$)]
    add node for closed set cl($\emptyset$) to H
    while Queue is not empty **do**
        $N \leftarrow$ first element of Queue, remove N from Queue
        for all minimal $N_i :=$ cl$(N \cup \{i\})$, where $i \in S \backslash N$ **do**
            if N_i does not occur as a node in H yet **then**
                add new node for closed set N_i to H
                add N_i to Queue
            end if
            add arc from N to N_i to H
        end for
    end while
    return H
end procedure



Benefits of This Approach

- Ganter's Algorithm is linear in the output size: edges and nodes. This is optimal.
- This is a very general and abstract framework, which is already implemented, for example in `polymake`.
- There are variants of Ganter's Algorithm, for example a linear enumeration of the closed sets.
- The method needs 'just' the implementation of a closure operator.

We saw several examples, but there are many more.

Generic Tropical Planes in 7 Dimensional Space

Speyer conjectured an upper bound for the number of cells.
Known: His bound holds and is achieved if it is a tropicalization of a linear space in characteristic 0.

Generic Tropical Planes in 7 Dimensional Space

Speyer conjectured an upper bound for the number of cells.
Known: His bound holds and is achieved if it is a tropicalization of a linear space in characteristic 0.

Theorem (Joswig, Hampe, S.)

Every generic tropical plane in $\mathbb{R}^8/\mathbb{R}\mathbf{1}$ *has one of four possible* f*-vectors:*

- *There are nine different combinatorial types of such planes, whose* f*-vector is* $(13, 55, 63)$ *and its bounded* f*-vector is* $(13, 15, 3)$*. Non of these is a tropicalization of a linear space in characteristic* 0*.*
- *There are 3013 different other combinatorial types, with* f*-vectors:*
  - $(13, 56, 64)$ *and bounded* f*-vector* $(13, 16, 4)$*.*
  - $(14, 58, 65)$ *and bounded* f*-vector* $(14, 18, 5)$*.*
  - $(15, 60, 66)$ *and bounded* f*-vector* $(15, 20, 6)$*.*