On the toric ideals of matroids of fixed rank

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    for B, B' and $b' \in B' \setminus B$ there is $b \in B \setminus B'$ such that $(B \setminus b) \cup b'$ is a basis

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- ... and by many other ways (circuits, flats, hyperplanes)

examples

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are bases. Then clearly $y_{B_1} y_{B_2} - y_{B_1'} y_{B_2'} \in I_M$.

White's conjectures

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For every matroid M, its toric ideal I_M is generated by quadratic binomials corresponding to symmetric exchanges.

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Question (Herzog, Hibi '02)

Is the base ring S_M/I_M Koszul?
Does the toric ideal I_M of a matroid M possess a quadratic Gröbner basis?

results for special classes

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- '14 L., Michałek strongly base orderable matroids

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$c(M) = \deg(I_M)r(M)|\mathfrak{B}|$

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White's conjecture is true 'for high degrees' w.r.t. the rank.

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Theorem (L. '16)

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for every rank r there exists a constant $c(r)$, such that if M is a matroid of rank r, then J_M, I_M agree starting from degree $c(r)$

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If M is a matroid of rank r, then its toric ideal I_M has a Gröbner basis of degree at most $2(r+3)!$.

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Corollary

Checking if White's conjecture is true for matroids of a fixed rank is decidable.

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3) $y_{D_1} \dots y_{D_{n-(r+3)!}} (y_{B_1} \dots y_{B_{(r+3)!}} - y_{B'_1} \dots y_{B'_{(r+3)!}}) \in I_M$

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5) use Ramsey-type result for blow-ups of bases

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Conjecture (a)

Complementary basis graph of a k-matroid is connected.

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Complementary basis graph of a k-matroid is connected.

Conjecture (b)

Let $k \geqslant 2$, and let M be a matroid of rank r on the ground set E of size $kr+1$. Suppose $x, y \in E$ are two elements such that both sets $E \setminus x$ and $E \setminus y$ can be partitioned into k pairwise disjoint bases. Then there exist partitions which share a common basis.

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Proposition

Conjectures (a) and (b) $\Rightarrow$ White's conjecture $\Rightarrow$ Conjecture (a).

Thank you!