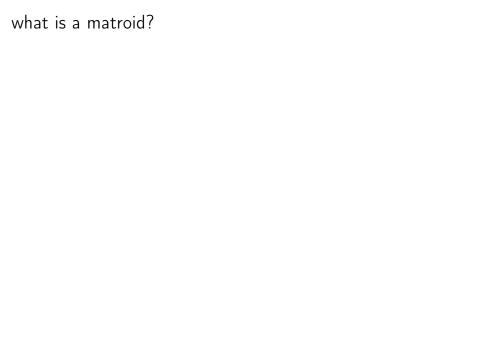
On the toric ideals of matroids of fixed rank

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Nice, 14th June 2017



what is a matroid?	
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independent sets

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- ... and by many other ways (circuits, flats, hyperplanes)

 \bullet representable matroid: E – a finite subset of a vector space

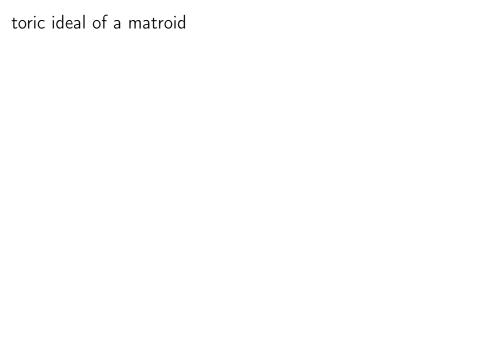
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are bases. Then clearly $y_{B_1}y_{B_2}-y_{B_1'}y_{B_2'}\in I_M$.

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For every matroid M, its toric ideal I_M is generated by quadratic binomials corresponding to symmetric exchanges.

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Question (Herzog, Hibi '02)

Is the base ring S_M/I_M Koszul? Does the toric ideal I_M of a matroid M possess a quadratic Gröbner basis?

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- '14 L., Michałek strongly base orderable matroids

general results

Theorem (L., Michałek '14)

White's conjecture is true 'up to saturation'.

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J_M = (y_{B_1}y_{B_2} - y_{B'_1}y_{B'_2} \text{ corresponding to symmetric exchange})
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conjecture: J_M = I_M
result: saturation of J_M with respect to \mathfrak{m} equals to I_M
conjecture: affine schemes Spec(S_M/I_M) = Spec(S_M/J_M)
result: projective schemes Proj(S_M/I_M) = Proj(S_M/J_M)
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c(M) = \deg(I_M)r(M)|\mathfrak{B}|
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If M is a matroid of rank r, then its toric ideal I_M has a Gröbner basis of degree at most 2(r+3)!.

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Corollary

Checking if White's conjecture is true for matroids of a fixed rank is decidable.

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Conjecture (a)

Complementary basis graph of a k-matroid is connected.

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Let $k \ge 2$, and let M be a matroid of rank r on the ground set E of size kr + 1. Suppose $x, y \in E$ are two elements such that both sets $E \setminus x$ and $E \setminus y$ can be partitioned into k pairwise disjoint bases. Then there exist partitions which share a common basis.

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Proposition

Conjectures (a) and (b) \Rightarrow White's conjecture \Rightarrow Conjecture (a).

Thank you!