An implicitization algorithm based on Chow Forms

For varieties of codimension larger than one

Clément Laroche

Ioannis Emiris

Christos Konaxis

ATHENA Research & Innovation Information Technologies

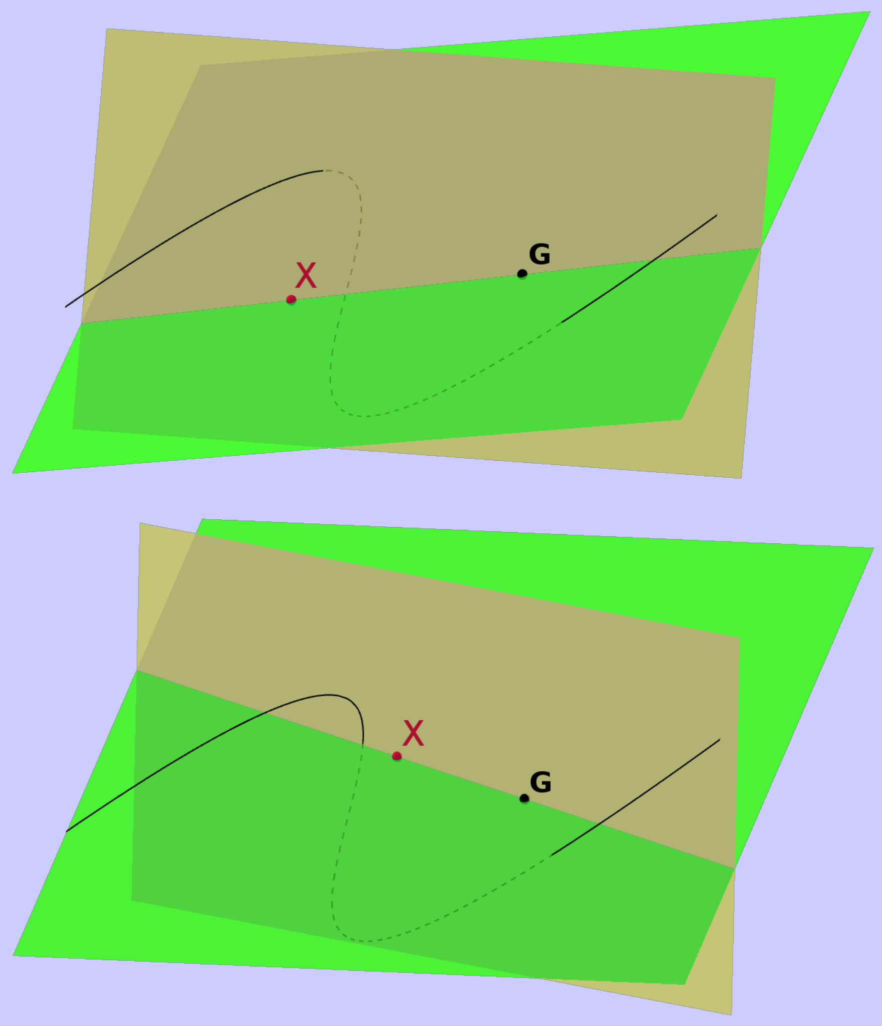
Marie Skłodowska-Curie Actions

HELLENIC REPUBLIC
National and Kapodistrian University of Athens

A Resultant Computation yields the Implicit Equation of a Conical Surface

Given a real parameterized variety V of dimension d and codimension $r > 1$ and points $G_1, \cdots, G_{r-1}$, we can detect if a point lies in one of the affine subspaces $\mathrm{aff}(\xi, G_1, \cdots, G_{r-1})$ with $\xi \in V$. The set of all such points is called the *conical hypersurface* of *apex* $(G_i)_i$ and *directrix* V.

Lemma. *Let $(H_i(t, X))_{1 \le i \le d+1}$ be the equations of random hyperplanes containing $G_1, \cdots, G_{r-1}$ and the parameterized point $\xi(t) \in V$.*
Then $\mathrm{Res}_t(H_1(t, X), \cdots, H_{d+1}(t, X))$ is an equation in X vanishing on the conical hypersurface.

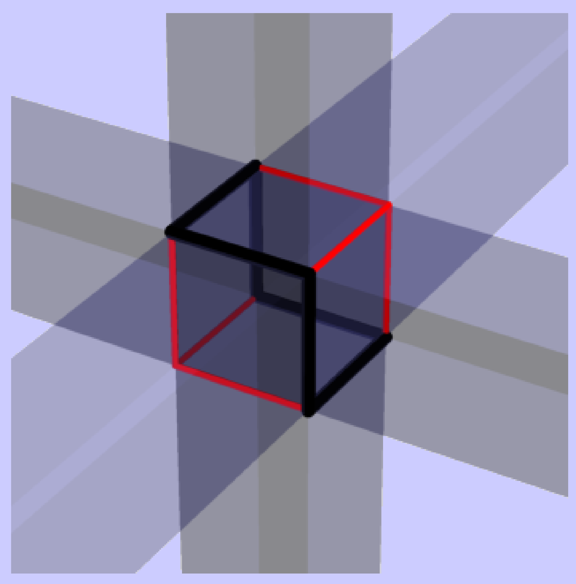


What is the Extraneous Factor?

In the Lemma above, $\mathrm{Res} = C(X).E(X)$ where $C(X)$ is the irreducible equation of the conical hypersurface and $E(X)$ is an extraneous factor.

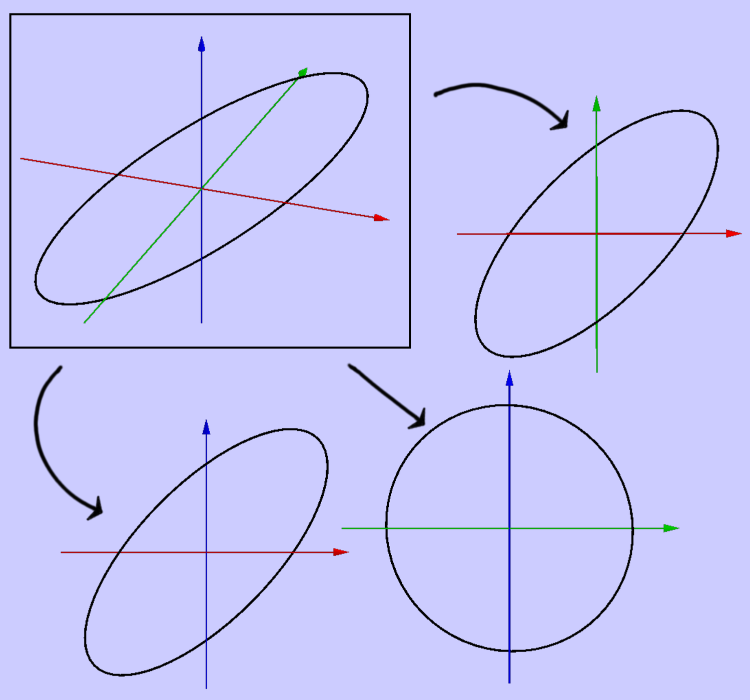
Property
$E(X)$ consists of a single irreducible polynomial of degree d raised to a certain power. Moreover, we have a closed formula for this polynomial.
In particular, we only need to compute a *linear* extraneous factor when $d = 1$.

How many Conical Hypersurfaces are Needed?



Since V is of codimension $r > 1$, one hypersurface is not enough to describe it.

For Space Curves, geometric and algebraic properties prove that 3 of these equations are enough.



How does the Algorithm Fare?

Algorithm	Degree	CPU Time	Output	Remarks
Chow Form	3	0.05s	3 equations of degree 3	The algorithm presented here
	5	0.14s	3 equations of degree 5	
	10	48s	3 equations of degree 10	
Ideal Elimination	3	0.03s	3 equations of degree 2	Use Maple's function `PolynomialIdeals[EliminationIdeal]`
	5	0.14s	5 equations of degree 4	
	10	2.48s	12 equations of degree 5 and 6	
Matrix Representation	3	0.42s	Matrix 2x3 of rank 2	Laurent Busé and Thang Luu Ba algorithm[2]
	5	0.83s	Matrix 4x7 of rank 4	Compute a matrix with a drop of rank property
	10	random	Matrix 7x11 of rank 7	`http://cgi.di.uoa.gr/~thanglb/MATRIXREP.mpl`

References

[1] J. Dalbec, B. Sturmfels, *Introduction to Chow Forms*, Springer, 1995

[2] L. Busé, *Implicit matrix representations of rational Bézier curves and surfaces*, Journal CAD, Special Issue 2013

Acknowledgements

This project has received funding from the European Union's Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement No 675789