# Imaginary Projections of Polynomials

Thorsten Jörgens

joint work with Thorsten Theobald and Timo de Wolff.

MEGA 2017, Nice

GOETHE UNIVERSITÄT FRANKFURT AM MAIN

# Introduction

Given: a polynomial $f \in \mathbb{C}[\mathbf{z}] := \mathbb{C}[z_1, \ldots, z_n]$.

# Introduction

Given: a polynomial $f \in \mathbb{C}[\mathbf{z}] := \mathbb{C}[z_1, \ldots, z_n]$.

Q: Understand the projections of the zero set

$$\mathcal{V}(f) = \{\mathbf{z} \in \mathbb{C}^n : f(\mathbf{z}) = 0\} \subseteq \mathbb{C}^n$$

onto these items:

- $\mathcal{V}(f) \to \mathbb{R}^n, \ \mathbf{z} \mapsto (|z_1|, \ldots, |z_n|)$,
- $\mathcal{V}(f) \cap (\mathbb{C}^*)^n \to \mathbb{R}^n, \ \mathbf{z} \mapsto (\log|z_1|, \ldots, \log|z_n|)$,
- $\mathcal{V}(f) \cap (\mathbb{C}^*)^n \to [0, 2\pi)^n, \ \mathbf{z} \mapsto (\arg z_1, \ldots, \arg z_n)$,
- $\mathcal{V}(f) \to \mathbb{R}^n, \ \mathbf{z} \mapsto (\operatorname{Im} z_1, \ldots, \operatorname{Im} z_n)$,

$\mathcal{V}(f) \to \mathbb{R}^n, \ \mathbf{z} \mapsto (\operatorname{val} z_1, \ldots, \operatorname{val} z_n)$ for polynomials over a field with a real valuation.

# Introduction

Given: a polynomial $f \in \mathbb{C}[\mathbf{z}] := \mathbb{C}[z_1, \dots, z_n]$.

Q: Understand the projections of the zero set

$$\mathcal{V}(f) = \{\mathbf{z} \in \mathbb{C}^n \,:\, f(\mathbf{z}) = 0\} \;\subseteq \mathbb{C}^n$$

onto these items:

- $\mathcal{V}(f) \to \mathbb{R}^n, \;\; \mathbf{z} \mapsto (|z_1|, \dots, |z_n|)$,

## Definition

- the *amoeba*
$$\mathcal{A}(f) \;=\; \{(\log|z_1|, \dots, \log|z_n|) \,:\, \mathbf{z} \in \mathcal{V}(f) \cap (\mathbb{C}^*)^n\},$$
- the *coamoeba*
$$co\mathcal{A}(f) \;=\; \{(\arg(z_1), \dots, \arg(z_n)) : \mathbf{z} \in \mathcal{V}(f) \cap (\mathbb{C}^*)^n\},$$

- $\mathcal{V}(f) \to \mathbb{R}^n, \;\; \mathbf{z} \mapsto \operatorname{Im}(z) = (\operatorname{Im} z_1, \dots, \operatorname{Im} z_n)$,

$\mathcal{V}(f) \to \mathbb{R}^n, \;\; \mathbf{z} \mapsto (\operatorname{val} z_1, \dots, \operatorname{val} z_n)$ for polynomials over a field with a real valuation.

# Introduction

Given: a polynomial $f \in \mathbb{C}[\mathbf{z}] := \mathbb{C}[z_1, \ldots, z_n]$.

Q: Understand the projections of the zero set

$$\mathcal{V}(f) = \{\mathbf{z} \in \mathbb{C}^n \, : \, f(\mathbf{z}) = 0\} \subseteq \mathbb{C}^n$$

onto these items:

- $\mathcal{V}(f) \to \mathbb{R}^n, \;\; \mathbf{z} \mapsto (|z_1|, \ldots, |z_n|)$,

## Definition

- the *amoeba*
$$\mathcal{A}(f) \;=\; \{(\log|z_1|, \ldots, \log|z_n|) \, : \, \mathbf{z} \in \mathcal{V}(f) \cap (\mathbb{C}^*)^n\},$$
- the *coamoeba*
$$co\mathcal{A}(f) \;=\; \{(\arg(z_1), \ldots, \arg(z_n)) : \mathbf{z} \in \mathcal{V}(f) \cap (\mathbb{C}^*)^n\},$$

## Definition

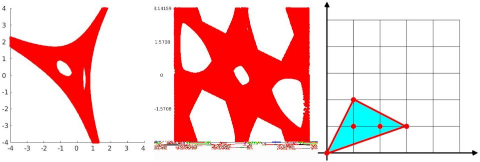
- the *imaginary projection*
$$\mathcal{I}(f) \;=\; \{(\mathrm{Im}(z_1), \ldots, \mathrm{Im}(z_n)) \, : \, \mathbf{z} \in \mathcal{V}(f)\}.$$

# Amoeba & coamoeba

## Definition

For $f \in \mathbb{C}[\mathbf{z}^{\pm 1}]$, the *amoeba* and *coamoeba* of $f$ are given by

$$\begin{aligned} \mathcal{A}(f) &= \{(\log|z_1|, \ldots, \log|z_n|) : \mathbf{z} \in \mathcal{V}(f) \cap (\mathbb{C}^*)^n\}, \\ co\mathcal{A}(f) &= \{(\arg(z_1), \ldots, \arg(z_n)) : \mathbf{z} \in \mathcal{V}(f) \cap (\mathbb{C}^*)^n\}. \end{aligned}$$




## Fact

*If* $\log_{\mathbb{C}}$ *is the complex logarithm, then* $\mathcal{A}(f) = \mathrm{Re} \circ \log_{\mathbb{C}} \mathcal{V}(f)$ *and* $co\mathcal{A}(f) = \mathrm{Im} \circ \log_{\mathbb{C}} \mathcal{V}(f)$, *where all map are understood component-wise.*

# The imaginary projection

## Definition

Given a polynomial $f \in \mathbb{C}[\mathbf{z}]$, define

$$\mathcal{I}(f) = \{\mathrm{Im}(\mathbf{z}) \,:\, \mathbf{z} \in \mathcal{V}(f)\} \,.$$

We call $\mathcal{I}(f)$ the *imaginary projection* of $f$.

Writing $z_j = x_j + iy_j$, the underlying projection is

$$\mathrm{Im} : \mathbb{R}^{2n} \to \mathbb{R}^n \,, \; (x_1, y_1, \ldots, x_n, y_n) \mapsto (y_1, \ldots, y_n)$$

- $\mathcal{I}(f)$ is a semialgebraic set.
- Components of the complement are convex.

# The imaginary projection

## Definition

Given a polynomial $f \in \mathbb{C}[\mathbf{z}]$, define

$$\mathcal{I}(f) = \{\mathrm{Im}(\mathbf{z}) \, : \, \mathbf{z} \in \mathcal{V}(f)\} \, .$$

We call $\mathcal{I}(f)$ the *imaginary projection* of $f$.

Writing $z_j = x_j + iy_j$, the underlying projection is

$$\mathrm{Im} : \mathbb{R}^{2n} \to \mathbb{R}^n \ , \ (x_1, y_1, \ldots, x_n, y_n) \mapsto (y_1, \ldots, y_n)$$

- $\mathcal{I}(f)$ is a semialgebraic set.
- Components of the complement are convex.

# The imaginary projection

## Definition

Given a polynomial $f \in \mathbb{C}[\mathbf{z}]$, define

$$\mathcal{I}(f) = \{\mathrm{Im}(\mathbf{z}) \,:\, \mathbf{z} \in \mathcal{V}(f)\} \,.$$

We call $\mathcal{I}(f)$ the *imaginary projection* of $f$.

Writing $z_j = x_j + iy_j$, the underlying projection is

$$\mathrm{Im} : \mathbb{R}^{2n} \to \mathbb{R}^n \,,\; (x_1, y_1, \ldots, x_n, y_n) \mapsto (y_1, \ldots, y_n)$$

- $\mathcal{I}(f)$ is a semialgebraic set.
- Components of the complement are convex.

# Motivation: stable polynomials

**Definition**

A polynomial $f \in \mathbb{C}[\mathbf{z}]$ is called *stable* if every root $\mathbf{z} = (z_1, \ldots, z_n)$ satisfies $\operatorname{Im}(z_j) \leq 0$ for some $j$.

- Marcus, Spielman, Srivastava: Proof of Kadison-Singer Conjecture, *Ann. Math.* 2015
- Marcus, Spielman, Srivastava: Existence of Ramanujan graphs, *Ann. Math.* 2015, FOCS 2013
- Borcea, Brändén: Proof of Johnson's Conjecture, *Duke Math. J.* 2008
- Gurvits: Simple proof of a generalization of van der Waerden's Conjecture, *Electron. J. Comb.* 2008

**Theorem**

*$f$ is stable if and only if $\mathcal{I}(f) \cap (\mathbb{R}_{>0})^n = \emptyset$.*

# Motivation: stable polynomials

## Definition

A polynomial $f \in \mathbb{C}[\mathbf{z}]$ is called *stable* if every root $\mathbf{z} = (z_1, \ldots, z_n)$ satisfies $\mathrm{Im}(z_j) \leq 0$ for some $j$.

- Marcus, Spielman, Srivastava: Proof of Kadison-Singer Conjecture, *Ann. Math.* 2015
- Marcus, Spielman, Srivastava: Existence of Ramanujan graphs, *Ann. Math.* 2015, FOCS 2013
- Borcea, Brändén: Proof of Johnson's Conjecture, *Duke Math. J.* 2008
- Gurvits: Simple proof of a generalization of van der Waerden's Conjecture, *Electron. J. Comb.* 2008

## Theorem

*$f$ is stable if and only if $\mathcal{I}(f) \cap (\mathbb{R}_{>0})^n = \emptyset$.*

# Motivation: stable polynomials

## Definition

A polynomial $f \in \mathbb{C}[\mathbf{z}]$ is called *stable* if every root $\mathbf{z} = (z_1, \ldots, z_n)$ satisfies $\text{Im}(z_j) \leq 0$ for some $j$.

- Marcus, Spielman, Srivastava: Proof of Kadison-Singer Conjecture, *Ann. Math.* 2015
- Marcus, Spielman, Srivastava: Existence of Ramanujan graphs, *Ann. Math.* 2015, FOCS 2013
- Borcea, Brändén: Proof of Johnson's Conjecture, *Duke Math. J.* 2008
- Gurvits: Simple proof of a generalization of van der Waerden's Conjecture, *Electron. J. Comb.* 2008

## Theorem

*$f$ is stable if and only if $\mathcal{I}(f) \cap (\mathbb{R}_{>0})^n = \emptyset$.*

# Affine-linear polynomials

## Example

Let $f(\mathbf{z}) = a_0 + \sum_{j=1}^{n} a_j z_j \in \mathbb{C}[\mathbf{z}]$ with $(a_1, \ldots, a_n) \in \mathbb{R}^n \setminus \{0\}$.

$$\Longrightarrow \mathcal{I}(f) = \mathcal{V}_{\mathbb{R}}(\mathrm{Im}(a_0) + \sum_{j=1}^{n} a_j y_j)$$

## Theorem

*Let $f(\mathbf{z}) = a_0 + \sum_{j=1}^{n} a_j z_j \in \mathbb{C}[\mathbf{z}]$ with $(a_1, \ldots, a_n) \neq 0$. Then:*

1. $$\mathcal{I}(f) = \begin{cases} \mathcal{V}_{\mathbb{R}}(\mathrm{Im}(a_0 e^{-i\varphi}) + \sum_{j=1}^{n} a_j e^{-i\varphi} y_j), & (a_1, \ldots, a_n) \in e^{i\varphi} \cdot \mathbb{R}^n \\ & \text{for some } \varphi \in [0, 2\pi), \\ \mathbb{R}^n, & \text{otherwise.} \end{cases}$$
2. *If all coefficients of $f$ are real, then $f$ is stable if and only if $a_1, \ldots, a_n \geq 0$ or $a_1, \ldots, a_n \leq 0$.*

# Affine-linear polynomials

## Example

Let $f(\mathbf{z}) = a_0 + \sum_{j=1}^n a_j z_j \in \mathbb{C}[\mathbf{z}]$ with $(a_1, \ldots, a_n) \in \mathbb{R}^n \setminus \{0\}$.

$$\Longrightarrow \mathcal{I}(f) = \mathcal{V}_{\mathbb{R}}\left(\mathrm{Im}(a_0) + \sum_{j=1}^n a_j y_j\right)$$

## Theorem

*Let* $f(\mathbf{z}) = a_0 + \sum_{j=1}^n a_j z_j \in \mathbb{C}[\mathbf{z}]$ *with* $(a_1, \ldots, a_n) \neq 0$*. Then:*

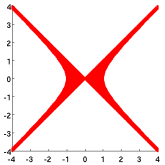
1. $$\mathcal{I}(f) = \begin{cases} \mathcal{V}_{\mathbb{R}}\left(\mathrm{Im}(a_0 e^{-i\varphi}) + \sum_{j=1}^n a_j e^{-i\varphi} y_j\right), & (a_1, \ldots, a_n) \in e^{i\varphi} \cdot \mathbb{R}^n \\ & \text{for some } \varphi \in [0, 2\pi), \\ \mathbb{R}^n, & \text{otherwise.} \end{cases}$$
2. *If all coefficients of* $f$ *are real, then* $f$ *is stable if and only if* $a_1, \ldots, a_n \geq 0$ *or* $a_1, \ldots, a_n \leq 0$*.*

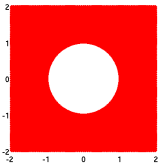
# Quadrics

**Theorem**

*For a quadratic polynomial $f \in \mathbb{R}[z_1, z_2]$, we have*

$$\mathcal{I}(f) = \begin{cases} \mathbb{R}^2 & \text{if } f \text{ is the ellipse } z_1^2 + z_2^2 - 1, \\ \{-1 \leq y_1^2 - y_2^2 < 0\} \cup \{0\} & \text{if } f \text{ is the hyperbola } z_1^2 - z_2^2 - 1, \\ \mathbb{R}^2 \setminus \{(0, y_2) : y_2 \neq 0\} & \text{if } f \text{ is the parabola } z_1^2 + z_2, \\ \{y_1^2 + y_2^2 - 1 \geq 0\} & \text{if } f \text{ is the empty set } z_1^2 + z_2^2 + 1. \end{cases}$$



# Quadrics

## Theorem

*For a quadratic polynomial $f \in \mathbb{R}[z_1, z_2]$, we have*

$$\mathcal{I}(f) = \begin{cases} \mathbb{R}^2 & \text{if } f \text{ is the ellipse } z_1^2 + z_2^2 - 1, \\ \{-1 \leq y_1^2 - y_2^2 < 0\} \cup \{0\} & \text{if } f \text{ is the hyperbola } z_1^2 - z_2^2 - 1, \\ \mathbb{R}^2 \setminus \{(0, y_2) : y_2 \neq 0\} & \text{if } f \text{ is the parabola } z_1^2 + z_2, \\ \{y_1^2 + y_2^2 - 1 \geq 0\} & \text{if } f \text{ is the empty set } z_1^2 + z_2^2 + 1. \end{cases}$$

And in the singular cases:

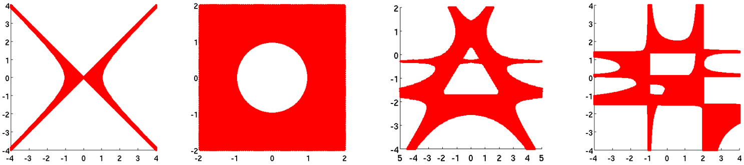
$$\mathcal{I}(f) = \begin{cases} \{y_1^2 - y_2^2 = 0\} & \text{if } f \text{ is a pair of crossing lines } z_1^2 - z_2^2 \\ \{0\} \times \mathbb{R} & \text{if } f \text{ are parallel lines } z_1^2 - 1 \text{ or a single line } z_1^2 \\ \mathbb{R}^2 & \text{if } f \text{ is an isolated point } z_1^2 + z_2^2 \\ \{\pm 1\} \times \mathbb{R} & \text{if } f \text{ is the empty set } z_1^2 + 1 \end{cases}$$

# Quadrics

## Theorem

*For a quadratic polynomial $f \in \mathbb{R}[z_1, z_2]$, we have*

$$\mathcal{I}(f) = \begin{cases} \mathbb{R}^2 & \text{if } f \text{ is the ellipse } z_1^2 + z_2^2 - 1, \\ \{-1 \leq y_1^2 - y_2^2 < 0\} \cup \{0\} & \text{if } f \text{ is the hyperbola } z_1^2 - z_2^2 - 1, \\ \mathbb{R}^2 \setminus \{(0, y_2) : y_2 \neq 0\} & \text{if } f \text{ is the parabola } z_1^2 + z_2, \\ \{y_1^2 + y_2^2 - 1 \geq 0\} & \text{if } f \text{ is the empty set } z_1^2 + z_2^2 + 1. \end{cases}$$



# Properties of the imaginary projection

For every non-constant $f \in \mathbb{C}[\mathbf{z}]$ the following holds:

- $\mathcal{I}(f)$ is a semialgebraic set.
- It is not always closed.
- For $n \geq 2$, it is always unbounded.
- If $f$ is irreducible, then $\mathcal{I}(f)$ is connected.
- For any $k \in \mathbb{Z}$, $k \geq 0$, there is a polynomial $f$ such that $\mathcal{I}(f)$ has exactly $k$ complement components.

**Theorem**

*For every polynomial $f \in \mathbb{C}[\mathbf{z}]$, all components of the complement of $\mathcal{I}(f)$ are convex. The number of these convex components is finite.*

# Proof: Convexity of the components in the complement

Let $C$ be one of the components. Define the holomorphic map

$$\psi : \mathbb{C}^n \to \mathbb{C}^n, \quad \mathbf{z} \mapsto \mathbf{z} \cdot e^{-i\frac{\pi}{2}},$$

(i.e., $\mathbf{x} + i\mathbf{y} \mapsto \mathbf{y} - i\mathbf{x}$). Moreover, let

$$C_\psi \;=\; \psi(\mathbb{R}^n + iC) \;=\; C - i\mathbb{R}^n \;=\; C + i\mathbb{R}^n.$$

- $C_\psi$ is a tubular region, i.e., for any $\mathbf{y} \in C_\psi \cap \mathbb{R}^n$ we have $\mathbf{y} + i\mathbf{x} \in C_\psi$ for all $\mathbf{x} \in \mathbb{R}^n$.
- The function $g : C_\psi \to \mathbb{C}$, $\mathbf{w} \mapsto \frac{1}{f(\psi(\mathbf{w}))}$ is holomorphic on $C_\psi$, and $C_\psi$ is the maximal tube with this property.

By Bochner's Tube Theorem, $g$ is holomorphic on conv $C_\psi$ (considered as set in $\mathbb{R}^{2n} \cong \mathbb{C}^n$). Due to the maximality of $C_\psi$, this implies the convexity of $C_\psi$ and thus of $C$.

Finiteness: as a consequence that $\mathcal{I}(f)$ is a semialgebraic set. □

# Proof: Convexity of the components in the complement

Let $C$ be one of the components. Define the holomorphic map

$$\psi : \mathbb{C}^n \to \mathbb{C}^n, \quad \mathbf{z} \mapsto \mathbf{z} \cdot e^{-i\frac{\pi}{2}},$$

(i.e., $\mathbf{x} + i\mathbf{y} \mapsto \mathbf{y} - i\mathbf{x}$). Moreover, let

$$C_\psi \ = \ \psi(\mathbb{R}^n + iC) \ = \ C - i\mathbb{R}^n \ = \ C + i\mathbb{R}^n.$$

- $C_\psi$ is a tubular region, i.e., for any $\mathbf{y} \in C_\psi \cap \mathbb{R}^n$ we have $\mathbf{y} + i\mathbf{x} \in C_\psi$ for all $\mathbf{x} \in \mathbb{R}^n$.
- The function $g : C_\psi \to \mathbb{C}$, $\mathbf{w} \mapsto \frac{1}{f(\psi(\mathbf{w}))}$ is holomorphic on $C_\psi$, and $C_\psi$ is the maximal tube with this property.

By Bochner's Tube Theorem, $g$ is holomorphic on conv $C_\psi$ (considered as set in $\mathbb{R}^{2n} \cong \mathbb{C}^n$). Due to the maximality of $C_\psi$, this implies the convexity of $C_\psi$ and thus of $C$.

Finiteness: as a consequence that $\mathcal{I}(f)$ is a semialgebraic set. □

# Proof: Convexity of the components in the complement

Let $C$ be one of the components. Define the holomorphic map

$$\psi : \mathbb{C}^n \to \mathbb{C}^n, \quad \mathbf{z} \mapsto \mathbf{z} \cdot e^{-i\frac{\pi}{2}},$$

(i.e., $\mathbf{x} + i\mathbf{y} \mapsto \mathbf{y} - i\mathbf{x}$). Moreover, let

$$C_\psi \;=\; \psi(\mathbb{R}^n + iC) \;=\; C - i\mathbb{R}^n \;=\; C + i\mathbb{R}^n.$$

- $C_\psi$ is a tubular region, i.e., for any $\mathbf{y} \in C_\psi \cap \mathbb{R}^n$ we have $\mathbf{y} + i\mathbf{x} \in C_\psi$ for all $\mathbf{x} \in \mathbb{R}^n$.
- The function $g : C_\psi \to \mathbb{C}$, $\mathbf{w} \mapsto \frac{1}{f(\psi(\mathbf{w}))}$ is holomorphic on $C_\psi$, and $C_\psi$ is the maximal tube with this property.

By Bochner's Tube Theorem, $g$ is holomorphic on conv $C_\psi$ (considered as set in $\mathbb{R}^{2n} \cong \mathbb{C}^n$). Due to the maximality of $C_\psi$, this implies the convexity of $C_\psi$ and thus of $C$.

Finiteness: as a consequence that $\mathcal{I}(f)$ is a semialgebraic set. $\square$

# Proof: Convexity of the components in the complement

Let $C$ be one of the components. Define the holomorphic map

$$\psi : \mathbb{C}^n \to \mathbb{C}^n, \quad \mathbf{z} \mapsto \mathbf{z} \cdot e^{-i\frac{\pi}{2}},$$

(i.e., $\mathbf{x} + i\mathbf{y} \mapsto \mathbf{y} - i\mathbf{x}$). Moreover, let

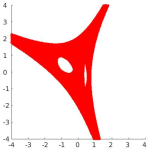
$$C_\psi \;=\; \psi(\mathbb{R}^n + iC) \;=\; C - i\mathbb{R}^n \;=\; C + i\mathbb{R}^n.$$

- $C_\psi$ is a tubular region, i.e., for any $\mathbf{y} \in C_\psi \cap \mathbb{R}^n$ we have $\mathbf{y} + i\mathbf{x} \in C_\psi$ for all $\mathbf{x} \in \mathbb{R}^n$.
- The function $g : C_\psi \to \mathbb{C}$, $\mathbf{w} \mapsto \frac{1}{f(\psi(\mathbf{w}))}$ is holomorphic on $C_\psi$, and $C_\psi$ is the maximal tube with this property.

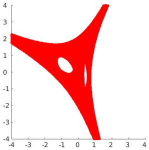
By Bochner's Tube Theorem, $g$ is holomorphic on conv $C_\psi$ (considered as set in $\mathbb{R}^{2n} \cong \mathbb{C}^n$). Due to the maximality of $C_\psi$, this implies the convexity of $C_\psi$ and thus of $C$.

Finiteness: as a consequence that $\mathcal{I}(f)$ is a semialgebraic set. □

# Behavior at infinity



# Behavior at infinity



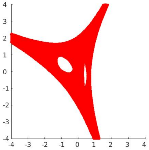

## Fact

*For the amoeba $\mathcal{A}(f)$ of a polynomial $f$, the logarithmic limit set*

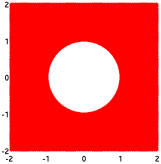
$$\mathcal{A}_\infty(f) = \lim_{r \to \infty} \left( \frac{1}{r} \mathcal{A}(f) \cap \mathbb{S}^{n-1} \right)$$

*is a spherical polyhedral complex.*

# Behavior at infinity




versus




## Fact

*For the amoeba $\mathcal{A}(f)$ of a polynomial $f$, the logarithmic limit set*

$$\mathcal{A}_\infty(f) \;=\; \lim_{r\to\infty}\left(\frac{1}{r}\mathcal{A}(f)\cap\mathbb{S}^{n-1}\right)$$

*is a spherical polyhedral complex.*

# Behavior at infinity

**Fact**

*For the amoeba $\mathcal{A}(f)$ of a polynomial $f$, the logarithmic limit set*

$$\mathcal{A}_\infty(f) \;=\; \lim_{r\to\infty}\left(\frac{1}{r}\mathcal{A}(f)\cap\mathbb{S}^{n-1}\right)$$

*is a spherical polyhedral complex.*

For the imaginary projection, the set

$$\mathcal{I}_\infty(f) \;=\; \lim_{r\to\infty}\left(\frac{1}{r}\mathcal{I}(f)\cap\mathbb{S}^{n-1}\right)$$

is not a spherical polyhedral complex in general.

# Behavior at infinity

## Fact

*For the amoeba $\mathcal{A}(f)$ of a polynomial $f$, the logarithmic limit set*

$$\mathcal{A}_\infty(f) \;=\; \lim_{r\to\infty}\left(\frac{1}{r}\mathcal{A}(f)\cap\mathbb{S}^{n-1}\right)$$

*is a spherical polyhedral complex.*

For the imaginary projection, the set

$$\mathcal{I}_\infty(f) \;=\; \lim_{r\to\infty}\left(\frac{1}{r}\mathcal{I}(f)\cap\mathbb{S}^{n-1}\right)$$

is not a spherical polyhedral complex in general.

## Example

Let $f(\mathbf{z}) = z_1^2 - \sum_{j=2}^n z_j^2 + 1$ with $n \geq 3$. Then,
$\mathcal{I}_\infty(f) \;=\; \left\{\mathbf{y}\in\mathbb{S}^{n-1} \,:\, y_1^2 \leq \frac{1}{2}\right\}$ is not a spherical polyhedral complex.

# Behavior at infinity

**Fact**

*For the amoeba $\mathcal{A}(f)$ of a polynomial $f$, the logarithmic limit set*

$$\mathcal{A}_\infty(f) \;=\; \lim_{r\to\infty}\left(\frac{1}{r}\mathcal{A}(f)\cap\mathbb{S}^{n-1}\right)$$

*is a spherical polyhedral complex.*

For the imaginary projection, the set

$$\mathcal{I}_\infty(f) \;=\; \lim_{r\to\infty}\left(\frac{1}{r}\mathcal{I}(f)\cap\mathbb{S}^{n-1}\right)$$

is not a spherical polyhedral complex in general.

**Definition**

We call a point $p \in \mathcal{I}_\infty(f)$ a *limit direction* of $\mathcal{I}(f)$.

# Imaginary projections and hyperbolic polynomials

## Definition

Let $f \in \mathbb{R}[\mathbf{z}]$ be homogeneous. Then $f$ is called *hyperbolic* in direction $\mathbf{e} \in \mathbb{R}^n$, if $f(\mathbf{e}) \neq 0$ and for every $\mathbf{x} \in \mathbb{R}^n$ the real function $t \mapsto f(\mathbf{x} + t\mathbf{e})$ has only real roots.

# Imaginary projections and hyperbolic polynomials

## Definition

Let $f \in \mathbb{R}[\mathbf{z}]$ be homogeneous. Then $f$ is called *hyperbolic* in direction $\mathbf{e} \in \mathbb{R}^n$, if $f(\mathbf{e}) \neq 0$ and for every $\mathbf{x} \in \mathbb{R}^n$ the real function $t \mapsto f(\mathbf{x} + t\mathbf{e})$ has only real roots.

## Definition

If $f$ is hyperbolic in direction $\mathbf{e} \in \mathbb{R}^n$, we call

$$C(\mathbf{e}) := \{\mathbf{x} \in \mathbb{R}^n \, : \, f(\mathbf{x} + t\mathbf{e}) = 0 \Rightarrow t < 0\}$$

the *hyperbolicity cone* of $f$ with respect to $\mathbf{e}$.

- $C(\mathbf{e})$ is open and convex (Gårding, 1959).
- $f$ is hyperbolic to every point $\mathbf{e}'$ in its hyperbolicity cone and $C(\mathbf{e}) = C(\mathbf{e}')$.

# Imaginary projections and hyperbolic polynomials

**Definition**

Let $f \in \mathbb{R}[\mathbf{z}]$ be homogeneous. Then $f$ is called *hyperbolic* in direction $\mathbf{e} \in \mathbb{R}^n$, if $f(\mathbf{e}) \neq 0$ and for every $\mathbf{x} \in \mathbb{R}^n$ the real function $t \mapsto f(\mathbf{x} + t\mathbf{e})$ has only real roots.

**Definition**

If $f$ is hyperbolic in direction $\mathbf{e} \in \mathbb{R}^n$, we call

$$C(\mathbf{e}) := \{\mathbf{x} \in \mathbb{R}^n \ : \ f(\mathbf{x} + t\mathbf{e}) = 0 \Rightarrow t < 0\}$$

the *hyperbolicity cone* of $f$ with respect to $\mathbf{e}$.

- $C(\mathbf{e})$ is open and convex (Gårding, 1959).
- $f$ is hyperbolic to every point $\mathbf{e}'$ in its hyperbolicity cone and $C(\mathbf{e}) = C(\mathbf{e}')$.

Application: Hyperbolic programming

$$\min \ \mathbf{c}^T\mathbf{x}$$

$$s.t. \ A\mathbf{x} = \mathbf{b}, \mathbf{x} \in \overline{C(\mathbf{e})}$$

# Imaginary projections and hyperbolic polynomials

## Definition

If $f$ is hyperbolic in direction $\mathbf{e} \in \mathbb{R}^n$, we call

$$C(\mathbf{e}) := \{\mathbf{x} \in \mathbb{R}^n \,:\, f(\mathbf{x} + t\mathbf{e}) = 0 \Rightarrow t < 0\}$$

the *hyperbolicity cone* of $f$ with respect to $\mathbf{e}$.

## Lemma

*Let $f \in \mathbb{R}[\mathbf{z}]$ be homogeneous of degree $d$. Then the number of hyperbolicity cones of $f$ is at most*

$$\begin{cases} 2^d & \text{for } d \leq n\,, \\ 2\sum_{k=0}^{n-1}\binom{d-1}{k} & \text{for } d > n\,. \end{cases}$$

*The maximum is attained if and only if $f$ is a product of independent linear polynomials.*

# Imaginary projections and hyperbolic polynomials

## Definition

If $f$ is hyperbolic in direction $\mathbf{e} \in \mathbb{R}^n$, we call

$$C(\mathbf{e}) := \{\mathbf{x} \in \mathbb{R}^n \,:\, f(\mathbf{x} + t\mathbf{e}) = 0 \Rightarrow t < 0\}$$

the *hyperbolicity cone* of $f$ with respect to $\mathbf{e}$.

## Theorem

*Let $f \in \mathbb{R}[\mathbf{z}]$ be homogeneous. Then the hyperbolicity cones of $f$ coincide with the complement components of $\mathcal{I}(f)$.*

# Imaginary projections and hyperbolic polynomials

## Definition

If $f$ is hyperbolic in direction $\mathbf{e} \in \mathbb{R}^n$, we call

$$C(\mathbf{e}) := \{\mathbf{x} \in \mathbb{R}^n \,:\, f(\mathbf{x} + t\mathbf{e}) = 0 \Rightarrow t < 0\}$$

the *hyperbolicity cone* of $f$ with respect to $\mathbf{e}$.

## Theorem

*Let $f \in \mathbb{R}[\mathbf{z}]$ be homogeneous. Then the hyperbolicity cones of $f$ coincide with the complement components of $\mathcal{I}(f)$.*

Proof. "$\Longleftarrow$" Let $f$ be hyperbolic with respect to $\mathbf{e} \in \mathbb{R}^n$.

Assume: $\mathbf{e} \in \mathcal{I}(f)$.

$\Longrightarrow$ For some $\mathbf{x} \in \mathbb{R}^n$, the imaginary unit $i$ is a root of the real function

$$t \mapsto f(\mathbf{x} + t\mathbf{e}).$$

Contradiction to the hyperbolicity of $f$.

# Imaginary projections and hyperbolic polynomials

## Definition

If $f$ is hyperbolic in direction $\mathbf{e} \in \mathbb{R}^n$, we call

$$C(\mathbf{e}) := \{\mathbf{x} \in \mathbb{R}^n \;:\; f(\mathbf{x} + t\mathbf{e}) = 0 \Rightarrow t < 0\}$$

the *hyperbolicity cone* of $f$ with respect to $\mathbf{e}$.

## Theorem

*Let $f \in \mathbb{R}[\mathbf{z}]$ be homogeneous. Then the hyperbolicity cones of $f$ coincide with the complement components of $\mathcal{I}(f)$.*

Proof. "$\Longrightarrow$" Let $\mathbf{e} \notin \mathcal{I}(f)$. Then $f(\mathbf{x} + i\mathbf{e}) \neq 0$ for all $\mathbf{x} \in \mathbb{R}^n$, so that

$$f(\mathbf{e}) \;=\; (1+i)^{-\deg f} f((1+i)\mathbf{e}) \;=\; (1+i)^{-\deg f} f(\mathbf{e} + i\mathbf{e}) \neq 0\,.$$

And if there exists an $\mathbf{x} \in \mathbb{R}^n$ such that $t \mapsto f(\mathbf{x} + t\mathbf{e})$ has a solution $a + ib$ with $b \neq 0$, then the homogeneous function $f$ would satisfy

$$f(\mathbf{x} + a\mathbf{e} + ib\mathbf{e}) \;=\; 0$$

in contradiction to $\mathbf{e} \notin \mathcal{I}(f)$. Hence, $f$ is hyperbolic with respect to $\mathbf{e}$. $\square$

For non-homogeneous polynomials:
Let $\text{in}(f)$ be the *initial form* of $f$, i.e., the sum of all terms with maximal total degree. Note that $\text{in}(f)(\mathbf{z}) = f_h(0, \mathbf{z})$.

**Proposition**

*For $f \in \mathbb{C}[\mathbf{z}]$, the sets of limit directions $\mathcal{I}_\infty(f)$ and $\mathcal{I}_\infty(\text{in}(f))$ coincide.*

**Proposition**

*For $f \in \mathbb{C}[\mathbf{z}]$, there is a bijection between the set of unbounded components of $\mathcal{I}(f)^c$ with full-dimensional recession cone and the hyperbolicity cones of $\text{in}(f)$.*

**Theorem**

*Let $f \in \mathbb{C}[\mathbf{z}]$ be a polynomial of degree $d$. Then the number of components with full-dimensional recession cone in $\mathcal{I}^c(f)$ is at most*

$$\begin{cases} 2^d & \text{for } d \leq n\,, \\ 2\sum_{k=0}^{n-1}\binom{d-1}{k} & \text{for } d > n\,. \end{cases}$$

For non-homogeneous polynomials:
Let $\mathrm{in}(f)$ be the *initial form* of $f$, i.e., the sum of all terms with maximal total degree. Note that $\mathrm{in}(f)(\mathbf{z}) = f_h(0, \mathbf{z})$.

**Proposition**

*For $f \in \mathbb{C}[\mathbf{z}]$, the sets of limit directions $\mathcal{I}_\infty(f)$ and $\mathcal{I}_\infty(\mathrm{in}(f))$ coincide.*

**Proposition**

*For $f \in \mathbb{C}[\mathbf{z}]$, there is a bijection between the set of unbounded components of $\mathcal{I}(f)^{\mathsf{c}}$ with full-dimensional recession cone and the hyperbolicity cones of $\mathrm{in}(f)$.*

**Theorem**

*Let $f \in \mathbb{C}[\mathbf{z}]$ be a polynomial of degree $d$. Then the number of components with full-dimensional recession cone in $\mathcal{I}^{\mathsf{c}}(f)$ is at most*

$$\begin{cases} 2^d & \text{for } d \leq n\,, \\ 2\sum_{k=0}^{n-1}\binom{d-1}{k} & \text{for } d > n\,. \end{cases}$$

For non-homogeneous polynomials:
Let $\text{in}(f)$ be the *initial form* of $f$, i.e., the sum of all terms with maximal total degree. Note that $\text{in}(f)(\mathbf{z}) = f_h(0, \mathbf{z})$.

**Proposition**

*For $f \in \mathbb{C}[\mathbf{z}]$, the sets of limit directions $\mathcal{I}_\infty(f)$ and $\mathcal{I}_\infty(\text{in}(f))$ coincide.*

**Proposition**

*For $f \in \mathbb{C}[\mathbf{z}]$, there is a bijection between the set of unbounded components of $\mathcal{I}(f)^c$ with full-dimensional recession cone and the hyperbolicity cones of $\text{in}(f)$.*

**Theorem**

*Let $f \in \mathbb{C}[\mathbf{z}]$ be a polynomial of degree $d$. Then the number of components with full-dimensional recession cone in $\mathcal{I}^c(f)$ is at most*

$$\begin{cases} 2^d & \text{for } d \leq n\,, \\ 2\sum_{k=0}^{n-1}\binom{d-1}{k} & \text{for } d > n\,. \end{cases}$$

**One open problem:**

- Let $f \in \mathbb{C}[\mathbf{z}]$ a non-homogeneous polynomial of total degree $d$ (or with Newton-polytope $P$). What is the maximal number of (bounded) complement components of $\mathcal{I}(f)$?

**One open problem:**

- Let $f \in \mathbb{C}[\mathbf{z}]$ a non-homogeneous polynomial of total degree $d$ (or with Newton-polytope $P$). What is the maximal number of (bounded) complement components of $\mathcal{I}(f)$?

Thank you for your attention!

**Articles:**

- T. Jörgens, T. Theobald and T. de Wolff. Imaginary projections of polynomials. ArXiV:1602.02008
- T. Jörgens and T. Theobald. Hyperbolicity Cones and Imaginary Projections. ArXiV:1703.04988