On semi-equivalence of generically-finite polynomial mappings

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Let $f, g : X \to Y$ be continuous mappings. We say that f is *topologically equivalent* to g if there exist homeomorphisms $\Phi : X \to X$ and $\Psi : Y \to Y$ such that $\Psi \circ f \circ \Phi = g$. Moreover, we say that f is topologically semi-equivalent to g if there exist open, dense subsets $U, V \subset X$ and homeomorphisms $\Phi : U \to V$ and $\Psi : Y \to Y$ such that $\Psi \circ f \circ \Phi|_U = g|_U$.

In the case $X = \mathbb{C}^n$ and $Y = \mathbb{C}$ René Thom stated a Conjecture that there are only finitely many topological types of polynomials $f : X \to Y$ of bounded degree. This Conjecture was confirmed by T. Fukuda. Also a more general problem was considered: how many topological types are there in the family $P(n, m, k)$ of polynomial mapping $f : \mathbb{C}^n \to \mathbb{C}^m$ of degree bounded by k? K. Aoki and H. Noguchi showed that there are only a finite number of topologically non-equivalent mappings in the family $P(2, 2, k)$. Finally I. Nakai showed that each familiy $P(n, m, k)$, where $n, m, k > 3$, contains infinitely many different topological types even if we consider only generically-finite mappings.

Hence the General Thom Conjecture is not true even for generically-finite mappings. However, we show in this paper that there are only a finite number of classes of topologically semi-equivalent generically-finite polynomial mappings $f : \mathbb{C}^n \to \mathbb{C}^m$ of a bounded (algebraic) degree.

Let us recall here, that a mapping $f : X \to Y$ is generically finite, if for general $x \in X$ the set $f^{-1}(f(x))$ is finite.

As a by product of our considerations we give a simple proof of the following interesting fact: for every n, m and k there are only a finite number of topological types of *proper* polynomial mappings $f : \mathbb{C}^n \to \mathbb{C}^m$ of (algebraic) degree bounded by k. Hence we can say that Thom Conjecture is true for proper polynomial mappings.

In fact we prove more: if X, Y are smooth affine irreducible varieties, then every algebraic family $\mathcal{F}$ of polynomial mappings from X to Y contains only a finite number of topologically non-semi-equivalent (non-equivalent) generically-finite (proper) mappings. Moreover, if a family $\mathcal{F}$ is irreducible, then two generic members of $\mathcal{F}$ are in the same equivalence class.

Let X, Z be affine irreducible varieties of the same dimension and assume that X is smooth. Let $f : X \to Z$ be a dominant polynomial mapping. It is well known that there is a Zariski open non-empty subset U of Z such that for every $x_1, x_2 \in U$ the fibers $f^{-1}(x_1), f^{-1}(x_2)$ have the same number $\mu(f)$ of points. We say that $\mu(f)$ is the topological degree of f. Recall the following (J):

Definition

Let X, Z be as above and let $f : X \to Z$ be a dominant polynomial mapping. We say that f *is finite at a point* $z \in Z$ if there exists an open neighborhood U of z such that the mapping $f_{|f^{-1}(U)} : f^{-1}(U) \to U$ is proper.

It is well-known that the set S_f of points at which the mapping f is not finite is either empty or it is a hypersurface (J). We say that S_f is *the set of non-properness* of f.

Definition

Let X be a smooth affine n-dimensional variety and let Z be an affine variety of the same dimension. Let $f : X \to Z$ be a generically finite dominant polynomial mapping of geometric degree $\mu(f)$. The bifurcation set of f is

$$B(f) = \{z \in Z : z \in Sing(Z) \ or \ \#f^{-1}(z) \neq \mu(f)\}.$$

Remark

The same definition makes sense for those continuous mapping $f : X \to Z$, for which we can define the topological degree $\mu(f)$ and singularities of Z. In particular if Z_1, Z_2 are affine algebraic varieties, $f : X \to Z_1$ is a dominant polynomial mapping and $\Phi : Z_1 \to Z_2$ is a homeomorphism which preserves singularities, then we can define $B(\Phi \circ f)$ as $\Phi(B(f))$. Moreover, the mapping $\Phi \circ f$ behaves topologically as an analytic covering.

We have the following theorem:

Theorem

Let X, Z be affine irreducible complex varieties of the same dimension and suppose X is smooth. Let $f : X \to Z$ be a polynomial dominant mapping. Then the set $B(f)$ is closed and $B(f) = K_0 \cup S_f \cup Sing(Z)$.

The proof of our result we begin with the following:

Lemma

Let $f : X^k \to Y^l$ *be a dominant polynomial mapping of affine irreducible varieties. There exists a Zariski open non-empty subset* $U \subset Y$ *such that for any* $y \in U$ *we have* $Sing(f^{-1}(y)) = f^{-1}(y) \cap Sing(X)$.

We have:

Lemma

Let X, Y be smooth complex irreducible algebraic varieties and $f : X \to Y$ a regular dominant mapping. Let $N \subset W \subset X$ be closed subvarieties of X. Then there exists a non-empty Zariski open subset $U \subset Y$ such that for every $y_1, y_2 \in U$ the triples $(f^{-1}(y_1), W \cap f^{-1}(y_1), N \cap f^{-1}(y_1))$ and $(f^{-1}(y_2), W \cap f^{-1}(y_2), N \cap f^{-1}(y_2))$ are homeomorphic.

We also need the following:

Definition

Let X, Y be smooth affine varieties. By a family of regular mappings $\mathcal{F}_M(X, Y, F) := \mathcal{F}$ we mean a regular mapping $F : M \times X \to Y$, where M is an algebraic variety. The members of a family $\mathcal{F}$ are the mappings $f_m : X \ni x \to F(m, x) \in Y$. Let

$$G : M \times X \ni (m, x) \mapsto (m, F(m, x)) \in Z = \overline{G(M \times X)} \subset M \times Y.$$

If G is generically finite, then by the topological degree $\mu(\mathcal{F})$ we mean the number $\mu(G)$. Otherwise we put $\mu(\mathcal{F}) = 0$.

Lemma

Let X, Y be smooth affine complex varieties. Let M be a smooth affine irreducible variety and let $\mathcal{F}$ be the family induced by a mapping $F : M \times X \to Y$, i.e.,
$\mathcal{F} = \{f_m : X \ni x \mapsto F(m,x) \in Y,\ m \in M\}$. Assume that $\mu(\mathcal{F}) > 0$. Take $Z = \overline{G(M \times X)}$ and put $Z_m = (m \times Y) \cap Z$. Then
1) there is an open non-empty subset $U_1 \subset M$ such that for every $m \in U_1$ we have $\mu(f_m) = \mu(\mathcal{F})$;
2) there is a non-empty open subset $U_2 \subset U_1$ such that for every $m \in U_2$ we have $\overline{f_m(X)} = Z_m := (m \times Y) \cap Z$ and $B(f_m) = B(G)_m := (m \times Y) \cap B(G)$;
3) there is a non-empty open subset $U_3 \subset U_2$ such that for every $m_1, m_2 \in U_3$ the pairs $(\overline{f_{m_1}(X)}, B(f_{m_1}))$ and $(\overline{f_{m_2}(X)}, B(f_{m_2}))$ are equivalent by means of a homeomorphism, i.e., there is a homeomorphism $\Psi : Y \to Y$ such that $\Psi(\overline{f_{m_1}(X)}) = \overline{f_{m_2}(X)}$ and $\Psi(B(f_{m_1})) = B(f_{m_2})$.

Now we are ready to prove our main result:

Theorem

Let X, Y be smooth affine irreducible varieties. Every algebraic family $\mathcal{F}$ of polynomial mappings from X to Y contains only a finite number of topologically non-semi-equivalent (non-equivalent) generically-finite (proper) mappings.

Our proof goes as follows. Let M be a smooth affine irreducible variety and let $\mathcal{F}$ be a family of polynomial mappings induced be a regular mapping $F : M \times X \to Y$, i.e.,
$\mathcal{F} := \{f_m : X \ni x \mapsto F(m,x) \in Y,\ m \in M\}$.

We showed that there exists a Zariski open, dense subset U of M such that

1) for every $m \in U$ we have $\mu(f_m) = \mu(\mathcal{F})$, where we treat f_m as a mapping $f_m : X \to Z_m := \overline{f_m(X)}$,

2) for every $m_1, m_2 \in U$ the pairs $(\overline{f_{m_1}(X)}, B(f_{m_1}))$ and $(\overline{f_{m_2}(X)}, B(f_{m_2}))$ are equivalent via a homeomorphism, i.e., there is a homeomorphism $\Psi : Y \to Y$ such that $\Psi(\overline{f_{m_1}(X)}) = \overline{f_{m_2}(X)}$ and $\Psi(B(f_{m_1})) = B(f_{m_2})$.

In particular the group $G = \pi_1(\overline{f_m(X)} \setminus B(f_m))$ does not depend on $m \in U$. Using elementary facts from the theory of topological coverings, we show that the number of topological semi-types (types) of generically-finite (proper) mappings in the family $\mathcal{F}_{|U}$ is bounded by the number of subgroups of G of index $\mu(\mathcal{F})$, hence it is finite. Then we conclude the proof by induction.

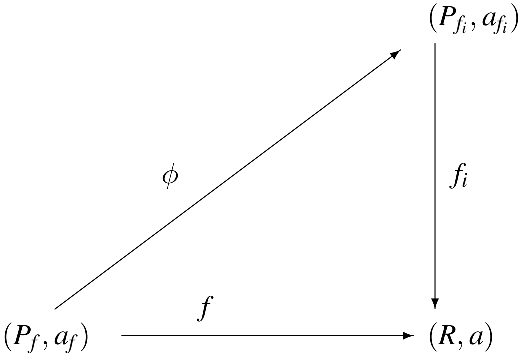
Let us recall the following result of M. Hall:

Lemma

Let G be a finitely generated group and let k be a natural number. Then there are only a finite number of subgroups $H \subset G$ such that $[G : H] = k$.

By Lemma of Hall there are only a finite number of subgroups $H_1, ..., H_r \subset G$ with index k. Choose generically-finite (proper) mappings $f_i = f'_{m_i} = \Psi_i \circ f_{m_i} : X \to Y$ such that $H_{f_i} = H_i$ (of course only if such a mapping f_i does exist). We show that every generically-finite (proper) mapping f'_m ($m \in U$) is semi-equivalent (equivalent) to one of mappings f_i.

Indeed, let $H_{f'_m} = H_{f_i}$ (here $f'_m = \Psi_m \circ f_m$). We show that $f'_m := f$ is equivalent to f_i. Let us consider two coverings $f : (P_f, a_f) \to (R, a)$ and $f_i : (P_{f_i}, a_{f_i}) \to (R, a)$. Since $f_*(\pi_1(P_f, a_f)) = f_{i*}(\pi_1(P_{f_i}, a_{f_i}))$ we can lift the covering f to a homeomorphism $\phi : P_f \to P_{f_i}$ such that following diagram commutes:


(P_{f_i}, a_{f_i})
ϕ
f_i
f
(P_f, a_f)
(R, a)

Hence for generically-finite mappings we have

$$(\Psi_i)^{-1} \circ \Psi_m \circ f_m \circ \phi^{-1}|_U = f_{m_i}|_U,$$

where $V = X \setminus f_m^{-1}(B(f_m))$ and $U = X \setminus f_{m_i}^{-1}(B(f_{m_i}))$. Hence f_m is semi-equivalent to f_{m_i}.

In the case of proper mappings we show additionally that the mapping ϕ can be extended to a continuous mapping Φ on the whole of X. Indeed, take a point $x \in f^{-1}(B)$ and let $y = f(x)$. The set $f_i^{-1}(y) = \{b_1, ..., b_s\}$ is finite. Take small open disjoint neighborhoods $W_i(r)$ of b_i, such that $W_i(r)$ shrinks to b_i as r tends to 0. We can choose an open neighborhood $V(r)$ of y so small that $f_i^{-1}(V(r)) \subset \bigcup_{j=1}^{s} W_i(r)$. Now take a small connected neighborhood $P_x(r)$ of x such that $f(P_x(r)) \subset V(r)$. The set $P_x(r) \setminus f^{-1}(B)$ is still connected and it is transformed by ϕ into one particular set $W_{i_0}(r)$. We take $\Phi(x) = b_{i_0}$. It is easy to see that the mapping Φ so defined is a continuous extension of ϕ. In fact $\phi(P_x(r) \setminus f^{-1}(B))$ shrinks to b_{i_0} if r goes to 0. Moreover, we still have $f = f_i \circ \Phi$.

In a similar way the mapping Λ determined by ϕ^{-1} is continuous. It is easy to see that $\Lambda \circ \Phi = \Phi \circ \Lambda = identity$, hence Φ is a homeomorphism. Consequently, the mapping $f_i \circ \Phi = \Psi_i \circ f_{m_i} \circ \Phi$ is equal to $f = \Psi_m \circ f_m$. Finally, we get

$$(\Psi_i)^{-1} \circ \Psi_m \circ f_m \circ \Phi^{-1} = f_{m_i}.$$

This means that the family $\mathcal{F}_{|U}$ contains only a finite number of topologically non-semi-equivalent (non-equivalent) generically-finite (proper) mappings.

Let $T = M \setminus U$. Hence dim $T <$ dim M. By the induction the family $\mathcal{F}_{|T}$ also contains only a finite number of topologically non-semi-equivalent (non-equivalent) generically-finite (proper) mappings. Consequently so does $\mathcal{F}$.

Corollary

There is only a finite number of topologically non-semi-equivalent (non-equivalent) generically-finite (proper) polynomial mappings $f : \mathbb{C}^n \to \mathbb{C}^m$ *of a bounded algebraic degree.*

Theorem

Let X, Y be smooth affine irreducible varieties. Let $\mathcal{F} : M \times X \to Y$ be an algebraic family of proper polynomial mappings from X to Y. Assume that M is an irreducible variety. Then there exists a Zariski open dense subset $U \subset M$ such that for every $m, m' \in U$ mappings f_m and $f_{m'}$ are topologically equivalent.

Theorem

Let X, Y be smooth affine irreducible varieties. Let $\mathcal{F} : M \times X \to Y$ be an algebraic family of generically-finite polynomial mappings from X to Y. Assume that M is an irreducible variety. Then there exists a Zariski open dense subset $U \subset M$ such that for every $m, m' \in U$ the mappings f_m and $f_{m'}$ are topologically semi-equivalent.

EXAMPLE. (J+Farnik). Let us see how looks topological classes in the case of quadratic polynomial mappings of the plane. We obtain a full classification of quadratic mappings of $\mathbb{C}^2$, with respect to the linear (hence also topological) equivalence. In particular we find a model of a generic mapping of $\Omega(2, 2)$.

Let $C(f), \Delta(f), \mu(f)$ denote : the set of critical points, the discriminant and topological degree. We have the following possibilities:

Generically-finite mappings:

(1) (the generic case) $f_1 = (x^2 + y, y^2 + x)$, $C(f_1) = \{4xy - 1 = 0\}$ is a hyperbola and $\Delta(f_1) = \{2^8x^2y^2 - 2^8x^3 - 2^8y^3 + 2^5 \cdot 9xy - 27 = 0\}$ is a reduced and irreducible curve with 3 cusps at points $f_1(\frac{\varepsilon}{2}, \frac{\varepsilon^2}{2})$, where $\varepsilon^3 = 1$, $\dim O(f_1) = 12$, $O(f_1)$ is an open and dense affine subvariety of $\Omega(2, 2)$, moreover $\chi(O(f_1)) = 0$ and $\mu(f_1) = 4$.

(2) $f_2 = (x^2 + y, xy)$, $C(f_2) = \{2x^2 = y\}$ is a parabola and $\Delta(f_2) = \{4x^3 = 27y^2\}$ is a cusp, $\dim O(f_2) = 11$, $O(f_2)$ is an affine subvariety of $\Omega(2, 2)$, $\mu(f_2) = 3$.

(3) $f_3 = (x^2 + y, y^2)$, $C(f_3) = \{4xy = 0\}$ is two intersecting lines and $\Delta(f_3) = \{y(y - x^2) = 0\}$ is the sum of a line and a parabola, $\dim O(f_3) = 11$, $\mu(f_3) = 4$.

(4) $f_4 = (x^2, y^2)$, $C(f_4) = \{4xy = 0\}$ is two intersecting lines and $\Delta(f_4) = \{xy = 0\}$ is also two intersecting lines, $\dim O(f_4) = 10$ and $\mu(f_4) = 4$.

(5) $f_5 = (x^2 - x, xy)$, $C(f_5) = \{2x^2 - x = 0\}$ is two parallel lines and $\Delta(f_5)$ is the sum of the line $x = -1/4$ and the line $x = 0$, $\dim O(f_5) = 10$, $\mu(f_5) = 2$ and f_5 is not proper.
(6) $f_6 = (x^2, xy)$, $C(f_6) = \{x^2 = 0\}$ is a double line and $\Delta(f_6)$ is the line $x = 0$, $\dim O(f_6) = 9$, $\mu(f_6) = 2$ and f_6 is not proper.
(7) $f_7 = (xy, x + y)$, $C(f_7) = \{y = x\}$ is a line and $\Delta(f_7) = \{4x - y^2 = 0\}$ is a parabola, $\dim O(f_7) = 10$ and $\mu(f_7) = 2$.
(7') $f_9 = (x^2, y)$, $C(f_9) = \{x = 0\}$ is a line and $\Delta(f_9) = \{x = 0\}$ is also a line, $\dim O(f_9) = 9$ and $\mu(f_9) = 2$.
(8) $f_8 = (x, xy)$, $C(f_8) = \{x = 0\}$ is a line and $\Delta(f_8)$ is the line $\{x = 0\}$, $\dim O(f_8) = 9$, $\mu(f_8) = 1$ and f_8 is not proper.
(9) $f_{10} = (x^2 + y, x)$, $\dim O(f_{10}) = 8$, $C(f_{10})$ is the empty set and f_{10} is an automorphism, $\mu(f_{10}) = 1$.
(9') $f_{12} = (x, y)$, $\dim O(f_{12}) = 6$, $C(f_{12})$ is the empty set and f_{12} is an automorphism, $\mu(f_{12}) = 1$.

Not generically-finite mappings:

(10) $f_{11}=(x^2,x)$, $\dim O(f_{11})=7$ and $C(f_{11})$ is the plane.
(10') $f_{14}=(x^2+y,0)$, $\dim O(f_{14})=7$ and $C(f_{14})$ is the plane.
(10") $f_{16}=(x,0)$, $\dim O(f_{16})=5$ and $C(f_{16})$ is the plane.
(11) $f_{13}=(xy,0)$, $\dim O(f_{13})=8$ and $C(f_{13})$ is the plane.
(12) $f_{15}=(x^2,0)$, $\dim O(f_{15})=6$ and $C(f_{15})$ is the plane.
(13) $f_{17}=(0,0)$, $\dim O(f_{17})=2$ and $C(f_{17})$ is the plane.

Note that mappings f_7 and f_9 are topologically (even algebraically) equivalent. Similarly mappings f_{10}, f_{12} and f_{11}, f_{14}, f_{16}. Hence we have 13 different topological types and 17 different linear types. Moreover we have 6 topologically non-equivalent classes of proper mappings.

THANK YOU FOR ATTENTION!