On Semiring Complexity of Schur Polynomials

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Elementary symmetric functions

$e_{k,n} = \sum_{1 \le i_1 < \cdots < i_k \le n} x_{i_1} \cdots x_{i_k}$, $0 \le k \le n$

Since

$\sum_{0 \le k \le n} e_{k,n} \cdot t^{n-k} = \prod_{1 \le i \le n} (t + x_i)$

the complexity

$C_{+,-,\times}(e_{0,n}, e_{1,n}, \ldots, e_{n,n}) \le O(n \cdot \log^2 n)$,

and moreover, the multiplicative complexity is bounded by $O(n \cdot \log n)$, since the complexity of multiplication of d-degree polynomials does not exceed $O(d \cdot \log d)$ due to the fast Fourier transform.

Theorem

*The (multiplicative) complexity of $e_{0,n}, e_{1,n}, \ldots, e_{n,n}$ is bounded from below by $\Omega(n \cdot \log n)$. (**V.Strassen, 1973**)*

If we don't allow subtraction then

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recursively applying the (Pascal triangle) formula

$e_{k,n} = e_{k,n-1} + x_n \cdot e_{k-1,n}$.

Question. Can one improve this bound?

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For a partition $\lambda = \{\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0\}$ define
$m_\lambda = \sum_{\pi \in S_n} x_{\pi(1)}^{\lambda_1} \cdots x_{\pi(n)}^{\lambda_n}$,
where the summation ranges over all the permutations π.

Golomb ruler is a sequence of integers $a_1 > \cdots > a_s$ such that the differences $a_i - a_j$, $i < j$ are pairwise distinct.

Theorem

*For a prime p the sequence $\lambda_{p-i+1} := 2pi + \{i^2 \bmod p\}$, $1 \leq i \leq p$, where $0 \leq i^2 \bmod p < p$, is a Golomb ruler (**Erdös-Turán, 1941**).*

Theorem

*If λ is a Golomb ruler then the complexity $C_{+,\times}(m_\lambda) \geq \Omega(c^n)$, $c > 1$. (**G.-Koshevoy**).*

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Lower bound on the complexity $C_{+,\times}$

For a polynomial P denote by $mon(P)$ the set of its monomials.

Lemma

*Let P be a homogeneous polynomial in n variables. If for any homogeneous polynomials Q, R such that $mon(P) \supset mon(Q \cdot R)$ and $\frac{1}{3} \cdot \deg(P) \leq \deg(Q), \deg(R) \leq \frac{2}{3} \cdot \deg(P)$ we have $|mon(P)| > c_1^n \cdot |mon(Q \cdot R)|$, $c_1 > 1$, then $C_{+,\times}(P) \geq \Omega(c_2^n)$, $c_2 > 1$. (**Schnorr, 1976, Valiant, 1980**)*

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Complete symmetric polynomials

$h_k(x_1,\ldots,x_n) := \sum_{1\le i_1\le\cdots\le i_k\le n} x_{i_1}\cdots x_{i_k}$

is homogeneous of degree k. Denote $\tilde{h}_k := h_k(x_1^2,\ldots,x_n^2)$. Then

$\sum_{0\le m<\infty} h_m\cdot t^m = \prod_{1\le i\le n}(1-x_i\cdot t)^{-1} =$

$\prod_{1\le i\le n}(1+x_i\cdot t)\cdot\prod_{1\le i\le n}(1-x_i^2\cdot t^2)^{-1} =$

$(\sum_{0\le a\le n} e_{a,n}\cdot t^a)\cdot(\sum_{0\le b<\infty}\tilde{h}_b\cdot t^{2b})$,

hence $h_m = \sum_{m-n\le 2b\le m} e_{m-2b,n}\cdot\tilde{h}_b$.

Denote the complexity $T(k) := C_{+,\times}(h_{k-n+1},\ldots,h_k)$ for a fixed n.

Then $T(k)\le T(\lceil k/2\rceil)+O(n^2)$ since

$C_{+,\times}(\tilde{h}_{\lceil k/2\rceil-n+1},\ldots,\tilde{h}_{\lceil k/2\rceil})\le T(\lceil k/2\rceil)+n$.

Corollary

$C_{+,\times}(h_k)\le O(n^2\cdot\log k)$ (**S.Fomin-G.-Nogneng-Schost**).

i. e. the algorithm is good for a fixed n and a large degree k.

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$\sum_{0 \le m < \infty} h_m \cdot t^m = \prod_{1 \le i \le n} (1 - x_i \cdot t)^{-1} =$
$\prod_{1 \le i \le n} (1 + x_i \cdot t) \cdot \prod_{1 \le i \le n} (1 - x_i^2 \cdot t^2)^{-1} =$
$(\sum_{0 \le a \le n} e_{a,n} \cdot t^a) \cdot (\sum_{0 \le b < \infty} \tilde{h}_b \cdot t^{2b})$,
hence $h_m = \sum_{m-n \le 2b \le m} e_{m-2b,n} \cdot \tilde{h}_b$.
Denote the complexity $T(k) := C_{+,\times}(h_{k-n+1}, \ldots, h_k)$ for a fixed n.
Then $T(k) \le T(\lceil k/2 \rceil) + O(n^2)$ since
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Corollary

$C_{+,-,\times}(s_\lambda) \leq O(n^3 \cdot |\lambda| \cdot \log^2 |\lambda|)$.

Complexity of Schur polynomials

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The proof is based on the cluster transformations.

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Semi-standard Young tableaux

The proof of the bound on $C_{+,\times}(s_\lambda)$ relies on the obtained bound on $C_{+,\times}(h_k)$ and on the formula expressing s_λ via the semi-standard Young tableaux T with the shape λ:
$T = (t_{i,j})$, $1 \leq t_{i,j} \leq n$, $1 \leq i \leq n$, $1 \leq j \leq \lambda_i$,
$t_{i+1,j} > t_{i,j}$ (strictly increasing down) and
$t_{i,j+1} \geq t_{i,j}$ (non-decreasing to the right).
Monomial $x^T := \prod_{i,j \in T} x_{t_{i,j}}$. Then $s_\lambda = \sum_T x^T$ where the summation ranges over all the semi-standard tableaux with the shape λ.

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