# Virtual refinements of the Vafa-Witten formula

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2. Moduli spaces of stable sheaves $M_S^H(r, c_1, c_2)$: $\{$rank $r$ coherent sheaves on $S$ with Chern classes $c_1, c_2\}$.

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We will study topological invariants of (2), using (1).

Interested in generating functions,

e.g. power series $\sum a_n x^n$, s.th. $a_n$ is the invariant of $S^{[n]}$

Formula for generating function reflects deep connection between different moduli spaces.

For vector bundle $E$ (or coherent sheaf) of rank $r$ on $X$ have
Chern classes $c_i(E) \in H^{2i}(X)$
measure how far $E$ is from being trivial i.e. $c_i(\mathcal{O}_X^{\oplus r}) = 0$
The Chern classes of $X$ are $c_i(X) = c_i(T_X)$.

$$\int_{[X]} c_n(X) = e(X) = \text{Euler number}$$

$$= \#\text{zeros of generic vector field} = \sum_{i=0}^{2n} (-1)^i \dim H^i(X)$$

The Euler number is one of the most basic and most important topological invariants.

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$H$ be an ample line bundle on $S$

$M_S^H(r, c_1, c_2) =$ moduli space of rank $r$ $H$-semistable sheaves on $S$ with Chern classes $c_1$, $c_2$

$\mathcal{E}$ semistable $\iff \frac{h^0(S,\mathcal{F}(n))}{\mathrm{rk}(\mathcal{F})} \leq \frac{h^0(S,\mathcal{E}(n))}{\mathrm{rk}(\mathcal{E})}$ for all $\mathcal{F}$ subsheaf of $\mathcal{E}$.

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$M_S^H(r, c_1, c_2)$ is usually singular, expected dimension

$$vd = 2rc_2 - (r-1)c_1^2 + r^2 - 1.$$

Here write $c_2 := \int_{[S]} c_2 \in \mathbb{Z}$, $c_1^2 := \int_{[S]} c_1^2 \in \mathbb{Z}$

We assume always that $p_g(S) = h^0(S, K_S) > 0$, $b_1(S) = 0$.

**Rank 1 case: Hilbert scheme of points**

$$S^{[n]} = \{\text{zero dimensional subschemes of length } n \text{ on } S\}$$

General pt $Z$ of $S^{[n]}$: $Z = p_1 \sqcup \ldots \sqcup p_n$ set of $n$ distinct pts of $S$
When points come together have nontrivial scheme structure, $Z = Z_1 \sqcup \ldots \sqcup Z_k$ such that $dim_{\mathbb{C}} \mathcal{O}_Z = \sum_{i=1}^{k} dim_{\mathbb{C}} \mathcal{O}_{Z_i} = n$.

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$M_S^H(1, L, c_2) = S^{[c_2]}$, via $Z \leftrightarrow I_Z \otimes \mathcal{O}(L)$. $I_Z$ ideal sheaf of $Z$.

**Euler numbers of Hilbert schemes:**

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**Theorem (G'90)**

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By physics arguments, 1994 Vafa and Witten gave explicit conjectural formula for the generating function for $e(M_S^H(2, L, n))$, in terms of modular forms.

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Write $K_S^2 = \int_{[S]} K_S^2 = \int_{[S]} c_1(S)^2$, let $\chi(\mathcal{O}_S)$ holomorphic Euler characteristic. Write in future $M_S^H(c_1, c_2) = M_S^H(2, c_1, c_2)$, and always

$$\mathrm{vd} = \mathrm{vd}_{M_S^H(c_1,c_2)} = 4c_2 - c_1^2 - 3\chi(\mathcal{O}_S)$$

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**Vafa-Witten conjecture**

$$\overline{\eta}(x) := x^{-1/24}\eta(x) = \prod_{n>0}(1 - x^n), \quad \theta_3^0(x) = \sum_{n\in\mathbb{Z}} x^{n^2}$$

$$\psi_S(x) := 8\left(\frac{1}{2\overline{\eta}(x^2)^{12}}\right)^{\chi(\mathcal{O}_S)}\left(\frac{2\overline{\eta}(x^4)^2}{\theta_3^0(x)}\right)^{K_S^2}$$

Then $e(M_S^H(c_1, c_2)) = \mathrm{Coeff}_{x^{\mathrm{vd}}}[\psi_S(x)]$.

Want to interpret, check and refine this formula.

$M = M_S^H(c_1, c_2)$ usually very singular
might have dimension different from $\mathrm{vd} = 4c_2 - c_1^2 - 3\chi(\mathcal{O}_S)$
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**Idea:** virtual Euler number $e^{\mathrm{vir}}(M)$ and all other virtual invariants of $M$ are invariant under deformation
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Virtual structure is used to define most invariants in modern enumerative geometry, e.g. Gromov-Witten, Donaldson invariants, Donaldson Thomas invariants, it forshadows Derived Algebraic Geometry.

**Perfect obstruction theory:**

Complex $E^\bullet = [E^{-1} \to E^0]$ of vb on $M$,

$T_M^{\mathrm{vir}} = [E_0] - [E_1] \in K^0(M), \quad E_i = (E^{-i})^\vee$

$\mathrm{vd} = \mathrm{rk}\, T_M^{\mathrm{vir}} = \mathrm{rk}(E_0) - \mathrm{rk}(E_1)$

virtual fundamental class $[M]^{\mathrm{vir}} \in H_{2\mathrm{vd}}(M)$

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### Conjecture

*The Vafa-Witten formula holds with $e(M_S^H(c_1, c_2))$ replaced by $e^{\mathrm{vir}}(M_S^H(c_1, c_2))$.*

**holomorphic Euler characteristic:**

$$\chi(X,V) := \sum_{i\geq 0}(-1)^i \dim H^i(X,V), \quad V \in K^0(X)$$

**$\chi_{-y}$-genus:**

$$\chi_{-y}(X) = \sum_{p,q}(-1)^{p+q} y^p h^{p,q}(X) = \sum_p (-y)^p \chi(X,\Omega_X^p)$$

alternating sum of Hodge numbers

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**Virtual $\chi_{-y}$-genus.** For $V \in K^0(M)$, put $\chi^{\mathrm{vir}}(M, V) := \chi(M, \mathcal{O}_M^{\mathrm{vir}} \otimes V)$. Let $\Omega_M^{\mathrm{vir}} := (T_M^{\mathrm{vir}})^\vee$.

$$\chi_{-y}^{\mathrm{vir}}(M) := y^{-\mathrm{vd}/2} \sum_p (-y)^p \chi^{\mathrm{vir}}(M, \Lambda^p \Omega_M^{\mathrm{vir}})$$

$\chi_{-1}^{\mathrm{vir}}(M) = e^{\mathrm{vir}}(M)$, so this is refinement of virtual Euler number

## Conjecture for virtual $\chi_{-y}$-genus:

$$\theta_3(x,y) := \sum_{n\in\mathbb{Z}} x^{n^2} y^n, \quad \overline{\eta}(x) = \prod_{n>0}(1-x^n)$$

$$\psi_S(x,y) := 8\left(\frac{1}{2\prod_{n>0}(1-x^{2n})^{10}(1-x^{2n}y)(1-x^{2n}/y)}\right)^{\chi(\mathcal{O}_S)} \cdot \left(\frac{2\overline{\eta}(x^4)^2}{\theta_3(x,y^{1/2})}\right)^{K_S^2}$$

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**Conjecture**

$\chi_{-y}^{\mathrm{vir}}(M_S^H(c_1,c_2)) = \mathrm{Coeff}_{x^{\mathrm{vd}}}[\psi_S(x,y)]$.

Specializes to our version of VW conjecture for $y=1$.

**Elliptic genus:** (Introduced by Witten, motivated by physics).
The elliptic genus is a refinement of the $\chi_{-y}$-genus.
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For vector bundle $E$ put

$$Ell_{z,y}(E) = y^{-\operatorname{rk}(E)/2} \bigotimes_{n\geq 1} \left(\Lambda_{-yz^{n-1}} E^\vee \otimes \Lambda_{-yz^n} E \otimes S_{z^n} E^\vee \otimes S_{z^n} E\right),$$

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$Ell(X) := \chi(X, Ell_{z,y}(T_X))$ elliptic genus.
$Ell^{\mathrm{vir}}(M) := \chi^{\mathrm{vir}}(M, Ell_{z,y}(T_M^{\mathrm{vir}}))$ virtual elliptic genus.
for $z = 0$ $Ell^{\mathrm{vir}}(M)$ specializes to $\chi_{-y}^{\mathrm{vir}}(M)$.

**DMVV formula** (conj. Dijkgraaf-Moore-Verlinde-Verlinde '97), (proof: Borisov-Libgober '00)
For

$$\phi(z,y)=\sum_{m,l}c_{m,l}y^lz^m\in\mathbb{Z}[y^{\pm1}][[z]]$$

put

$$L(\phi):=\prod_{n>0}\prod_{m,l}(1-x^ny^lz^m)^{c_{nm,l}}$$

$L$ Borcherds lift, Jacobi form $\mapsto$ Siegel modular form (something like modular form in three variables).
Then

$$\sum_{n\geq0}Ell(S^{[n]})x^n=\frac{1}{L(Ell(S))}=\left(\frac{1}{L(24\phi_2)}\text{ for }S=K3\right).$$

$$G_{1,0}(z,y) = -\frac{1}{2}\frac{y+1}{y-1} + \sum_{n>0}\sum_{d|n}(y^d - y^{-d})z^n$$

$$G_{2,0}(z,y) = \frac{1}{12}\frac{y^2+10y+1}{(y-1)^2} + \sum_{n>0}\sum_{d|n} d(y^d - 2 + y^{-d})z^n$$

$$G_{3,0}(z,y) = -\frac{y^2+y}{(y-1)^3} + \sum_{n>0}\sum_{d|n} d^2(y^d - y^{-d})z^n$$

$$\phi_i(z,y) := G_{i,0}(z,y)\frac{\theta(z,y)^i}{\eta(z)^{3i}}, \quad \theta(z,y) = \sum_{n\in\mathbb{Z}}(-1)^n z^{(n+1/2)^2/2} y^{n+1/2},$$

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## Conjecture

$$Ell^{\mathrm{vir}}(M_S^H(c_1,c_2)) = \mathrm{Coeff}_{x^{vd}}\left[8\left(\frac{1}{2}\frac{1}{L(12\phi_2)|_{x=x^2}}\right)^{\chi(\mathcal{O}_S)}\right.$$

$$\left.\cdot\left(\frac{L(2\phi_1^{\mathrm{ev}}|_{z=z^{1/2}})L(\phi_1|_{z=z^2,y=y^2})}{L(2\phi_1^2)}\Big|_{x=x^2}\cdot\frac{2L(2\phi_1\phi_3)|_{x=x^4}}{L(2\phi_1)}\right)^{K_S^2}\right].$$

**Final generalization: the cobordism class:**
Two complex manifolds $M$, $N$ have the same cobordism class
$\{M\} = \{N\}$
if they have the same Chern numbers:

$$\int_{[M]} c_{i_1}(M) \cdots c_{i_k}(M) = \int_{[N]} c_{i_1}(N) \cdots c_{i_k}(N) \quad \forall_{k, i_1, \ldots, i_k}$$

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Cobordism classes of complex manifolds generate a ring
$R = \sum_n R_n$ (graded by dimension)
$\{M\}\{N\} = \{M \times N\}, \quad \{M\} + \{N\} = \{M \sqcup N\}$
In fact

$$R \otimes \mathbb{Q} = \mathbb{Q}[\{\mathbb{P}^1\}, \{\mathbb{P}^2\}, \{\mathbb{P}^3\}, \ldots]$$

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For $M = M_S^H(c_1, c_2)$ let $\{M\}^{\mathrm{vir}}$ be the virtual cobordism class given by the

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## Conjecture

*There is a power series $P(x) = 1 + \sum_{n>0} P_n x^n$, with $P_n \in R_n$, s.th.*

$$\{M_S^H(c_1, c_2)\}^{\mathrm{vir}} = \mathrm{Coeff}_{x^{\mathrm{vd}}}\left[8\left(\frac{1}{4}\sum_{n\geq 0}\{K3^{[n]}\}x^{2n}\right)^{\chi(\mathcal{O}_S)/2}(2P(x))^{K_S^2}\right].$$

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Compute intersection numbers on $M = M_S^H(c_1, c_2)$ in terms of intersection numbers on Hilbert scheme of points.

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On $S \times M$ have $\mathcal{E}$ universal sheaf
i.e. if $[E] \in M$ corresponds to a sheaf $E$ on $S$ then $\mathcal{E}|_{S\times[E]} = E$.
For $\alpha \in H^k(S)$, put

$$\tau_i(\alpha) := \pi_{M_*}(c_i(\mathcal{E})\pi_S^*(\alpha)) \in H^{2i-4+k}(S)$$

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Let $P(\mathcal{E})$ be any polynomial in the $\tau_i(\alpha)$
Mochizuki's formula expresses $\int_{[M]^{\mathrm{vir}}} P(\mathcal{E})$ in terms of intersec. numbers on $S^{[n_1]} \times S^{[n_2]}$, and Seiberg-Witten invariants.

$e^{\mathrm{vir}}(M)$, $\chi_{-y}^{\mathrm{vir}}(M)$ $Ell^{\mathrm{vir}}(M)$ and $\{M\}^{\mathrm{vir}}$ can all be expressed as $\int_{[M]^{\mathrm{vir}}} P(\mathcal{E})$, so can reduce computation to Hilbert schemes.

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For $\chi_{-y}^{\mathrm{vir}}(M)$ $Ell^{\mathrm{vir}}(M)$ use **virtual Riemann-Roch formula**

**Theorem (Fantechi-G.)**

*For $V \in K^0(M)$ have*

$$\chi^{\mathrm{vir}}(M, V) = \int_{[M]^{\mathrm{vir}}} \mathrm{ch}(V)\mathrm{td}(T_M^{\mathrm{vir}}).$$

**Seiberg-Witten invariants:**
differentiable invariants of differentiable 4-manifolds
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If $b_1(S) = 0$, $p_g(S) > 0$ and $|K_S|$ contains smooth connected curve, then SW cl. of $S$ are $0, K_S$ with

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**Seiberg-Witten invariants:**
differentiable invariants of differentiable 4-manifolds
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This is the reason for our assumption that $|K_S|$ contains smooth connected curve, otherwise our results are more complicated.

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Two universal sheaves: Let $a\in \mathrm{Pic}(S)$

1. $\mathcal{I}_i(a)$ sheaf on $S\times S^{[n_1]}\times S^{[n_2]}$ with $\mathcal{I}_i(a)|_{S\times(Z_1,Z_2)}=I_{Z_i}\otimes a$
2. $\mathcal{O}_i(a)$, vector bundle of rank $n_i$ on $S^{[n_1]}\times S^{[n_2]}$, with fibre $\mathcal{O}_i(a)(Z_1,Z_2)=H^0(\mathcal{O}_{Z_i}\otimes a)$

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For a vector bundle $E$ of rank $r$ and variable $s$ put

$$c_i(E \otimes s) = \sum_{k=0}^{i} \binom{r-i}{k} s^{i-k} c_k(E), \quad Eu(E) = c_r(E)$$

For sheaves $\mathcal{E}_1, \mathcal{E}_2$ on $S \times S^{[n_1]} \times S^{[n_2]}$ put

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$$\Psi(a_1, a_2, n_1, n_2, s) = \frac{P\big(\mathcal{I}_1(a_1) \otimes s^{-1} \oplus \mathcal{I}_2(a_2) \otimes s\big) Eu(\mathcal{O}_1(a_1)) Eu(\mathcal{O}_2(a_2) \otimes s^2)}{Q(\mathcal{I}_1(a_1) \otimes s^{-1}, \mathcal{I}_2(a_2) \otimes s) \cdot (2s)^{n_1+n_2-\chi(\mathcal{O}_S)}}$$

$$A(a_1, a_2, c_2, s) = \sum_{n_1+n_2=c_2-a_1a_2} \int_{S^{[n_1]} \times S^{[n_2]}} \Psi(a_1, a_2, n_1, n_2, s)$$

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**Theorem (Mochizuki)**

*Assume $\chi(E) > 0$ for $E \in M_H^S(c_1, c_2)$. Then*

$$\int_{[M_S^H(c_1,c_2)]^{\mathrm{vir}}} P(\mathcal{E}) = \sum_{\substack{c_1=a_1+a_2 \\ a_1H<a_2H}} SW(a_1)\mathrm{Coeff}_{s^0} A(a_1, a_2, c_2, s)$$

Take now for $P(\mathcal{E}) = c_{\mathrm{vd}}(T_M^{\mathrm{vir}})$ (works the same for the others)
Put

$$Z_S(a_1, a_2, s, q) = \sum_{n_1, n_2 \geq 0} \int_{S^{[n_1]} \times S^{[n_2]}} A(a_1, a_2, a_1 a_2 + n_1 + n_2, s) q^{n_1 + n_2}$$

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## Proposition

*There exist univ. functions*

$$A_1(s,q), \ldots, A_7(s,q) \in \mathbb{Q}[s, s^{-1}][[q]]$$

*s.th.* $\forall_{S, a_1, a_2}$

$$Z_S(a_1, a_2, s, q) = F_0(a_1, a_2, s) A_1^{a_1^2} A_2^{a_1 a_2} A_3^{a_2^2} A_4^{a_1 K_S} A_5^{a_2 K_S} A_6^{K_S^2} A_7^{\chi(\mathcal{O}_S)},$$

*(where $F_0(a_1, a_2, s)$ is some explicit elementary function).*

**Proof:** Modification of an argument of Elllingsrud-G-Lehn: "Intersection numbers of universal sheaves on $S^{[n]}$ are universal polynomials in intersection numbers on $S$".

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**Reason:** Untersection numbers on $S^{[n]}$ computed inductively: $Z_n(S) := \{(x, Z) \in S \times S^{[n]} | x \in Z\}$ universal subscheme Blowup of $S \times S^{[n]}$ along $Z_n(S)$ is

$$S^{[n,n+1]} := \{(Z, W) \in S^{[n]} \times S^{[n+1]} \mid Z \in W\}$$

This allows to compute intersection numbers of $S^{[n+1]}$ in terms of inters. numbers on $S$ and $S^{[n]}$, and conclude by induction.

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This gives:

$$\mathrm{Coeff}_{q^k s^l} Z_S(a_1, a_2, s, q) = P_{k,l}(a_1^2, a_1 a_2, a_2^2, a_1 K_S, a_1 K_S, K_S^2, \chi(O_S))$$

for some polynomial $P_{k,l}$ depending only on $k, l$.
For the multiplicativity use additional tricks.

$A_1(s,q), \ldots A_7(s,q)$ are determined by value of $Z_S(a_1, a_2, s, q)$ for 7 triples $(S, a_1, a_2)$ ($S$ surface, $a_1, a_2 \in \mathrm{Pic}(S)$) s.th. corresponding 7-tuples $(a_1^2, a_1 a_2, a_2^2, a_1 K_S, a_1 K_S, K_S^2, \chi(O_S))$ are linearly independent

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We take

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In this case $S$ is a smooth toric, i.e. have an action of $T=\mathbb{C}^*\times\mathbb{C}^*$ with finitely many fixpoints,
Action of $T$ lifts to action on $S^{[n]}$ still with finitely many fixpoints described by partitions, compute by equivariant localization.
This computes $Z_S(a_1,a_2,s,q)$ in terms of combinatorics of partitions.

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This confirms conjectures for K3 surfaces, their blowups, elliptic surfaces, double covers of $\mathbb{P}^2$ and rational ruled surfaces, complete intersections, for $\mathrm{vd}(M)$ smaller than roughly $\frac{3}{2}$ times the power of $q$.

Let $X$ be a smooth projective variety with action of $T = \mathbb{C}^* \times \mathbb{C}^*$ with finitely many fixpoints, $p_1, \ldots, p_l$
Let $E$ be equivariant vector bundle of rank $r$ on $X$.

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Fibre $E(p_i)$ of $X$ at fixp. $p_i$ has basis of eigenvect. for $T$-action $E(p_i) = \bigoplus_{k=1}^r \mathbb{C} v_i$, with action $(t_1, t_2) \cdot v_i = t_1^{n_i} t_2^{m_i} v_i$, $n_i, m_i \in \mathbb{Z}$.

$$c^T(E(p_i)) = (1 + c_1^T(E(p_i)) + \ldots + c_r^T(E(p_i)) = \prod_{i=1}^r (1 + n_i\epsilon_1 + m_i\epsilon_2) \in \mathbb{Z}[\epsilon_1, \epsilon_2].$$

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Let $E_1, \ldots, E_s \in K_T^0(X)$, let $P([c_i(E_j)]_{i,j})$ be a polynomial in the Chern classes of these bundles, of degree $d = \dim(X)$.

**Theorem (Bott residue formula)**

$$\int_{[X]} P([c_i(E_j)]_{i,j}) = \sum_{k=1}^{l} \frac{P([c_i^T(E_j(p_k)]_{i,j})}{c_d^T(T_X(p_k))}$$

*(does not depend on $\epsilon_1, \epsilon_2$)*

For simplicity $S = \mathbb{P}^2$. $T = \mathbb{C}^* \times \mathbb{C}^*$ acts on $\mathbb{P}^2$ by

$$(t_1, t_2) \cdot (X_0 : X_1 : X_2) = (X_0 : t_1 X_1 : t_2 X_2)$$

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Easy: $Z$ is $T$-invariant $\iff I_Z \in k[x, y]$ is gen. by monomials
Can write

$$I_Z = (y^{n_0}, xy^{n_1}, ...., x^r y^{n_r}, x^{r+1}) \quad (n_0, \ldots, n_r) \text{ partition of } n$$

Fixpoints on $(\mathbb{P}^2)^{[n]}$ are in bijections with triples $(P_0, P_1, P_2)$ of partitions, adding up to $n$.

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Let e.g. $Z = Z_0$, $I_Z = (y^4, xy^2, x^2y, x^3)$
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Thus basis of eigenvectors of fibre for $T$ action is

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Thus for a general $\mathbb{C}^* \hookrightarrow T; t \mapsto (t^a, t^b)$ get

$$c^T(\mathcal{O}^{[n]}(Z)) = (1+\epsilon_2)(1+2\epsilon_2)(1+3\epsilon_2)(1+\epsilon_1)(1+\epsilon_1+\epsilon_2)(1+2\epsilon_1).$$

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$$\begin{aligned} \text{Euler number} &\leftrightarrow \text{4 dim. version} \\ \chi_{-y}\text{-genus} &\leftrightarrow \text{5 dim. comp. on a circle} \\ \text{Elliptic-genus} &\leftrightarrow \text{6 dim. comp. on elliptic curve} \end{aligned}$$