Extraction of tori from minimal point sets

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Extraction of geometric primitives from scanned 3D point clouds is important in reverse engineering.

Basic types of elementary shapes are: planes, spheres, cylinders, cones or tori.

RANSAC method “judges” if a primitive extracted from a small set of points is relevant to the full point cloud.

It is important to compute a shape through the smallest possible number of points.

Mixed data: some points with normals and some other points without normals.
CAD models

Extraction of tori from minimal point sets
A plane (resp. a sphere) is defined by 3 (resp. 4) parameters. It is also uniquely defined by 3 (resp. 4) of its points chosen generically, or by a point with its normal (resp. plus another point).

A cylinder (resp. a cone, resp a torus) is defined by 5 (resp. 6, resp. 7) parameters/unknowns. But it is not uniquely defined by 5 (resp.6, resp. 7) of its points chosen generically.

Our work studies and quantifies these ambiguities.
First perform a change of coordinates, in order to fix 6 input values. E.g. let $P_1$ at the origin, its normal $n_1$ on the $z$–axis and choose the abscissa of $P_2$ equal to 0.

Choose a set of parameters/unknowns with a "geometric" meaning, for constructing the surface.

Translate our construction into algebraic equations.

Provide formulas or efficient algorithms (resultants) to compute the parameters.

We consider general cases or a selection of special cases, and then report experiments with random data.
Figure: Configurations of points with four interpolating smooth tori.
Our main result

Theorem.

Generically, there exist at most eight non-degenerated tori that interpolate three distinct points, two of them being oriented.

This bound (eight) is reached, even when we restrict to real smooth tori (which is the useful practical setting).
A torus $T$ is defined by a skeletal circle $C$ and a radius $r > 0$.

- We consider two oriented points $p_1 := (0, 0, 0)$ and $N_1 = k := (0, 0, 1)$; $p_2 = (0, y_2, z_2)$ and $N_2 = N = (l, m, n)$; and we assume $k \wedge N \neq 0$.

- Call $C$ the skeletal circle, and let $r$ be the radius of a sectional circle.

- Assume $A_1 = p_1 + rk$ belongs to $C$, and $A_2 = p_2 + rN$ (with the same sign) also belongs to $C$.

**Strategy:** We will see that (generically) $C$ is uniquely determined by $(A_1, N_1)$ and $(A_2, N_2)$.

Then the equation of $T$ is parameterized by $r$. Expressing that the third point $p_3$ belongs to the torus $T$, provides the solution.
We suppose that two distinct points $A_1$ and $A_2$ and two vectors $N_1$ and $N_2$ are given.

- Consider the bisecting plane $\Pi$ of the segment $[A_1A_2]$.
- For $i = 1, 2$, $\Omega_i$ the intersection point between $\Pi$ and the line passing through $A_i$ and parallel to $N_i$. ($\Omega_i$ is possibly at infinity.)

**Proposition:**
If $\Omega_1$ and $\Omega_2$ are not both infinite, and $\Omega_1 \neq \Omega_2$, then there is exactly one fitting circle $C$. 
Fitting a 3D circle

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For $i = 1, 2$, $\Omega_i := A_i + \lambda_i N_i$.

Let

$$\lambda_1 = \frac{\|A_1A_2\|^2}{2N_1 \cdot A_1A_2}, \quad \lambda_2 = \frac{\|A_1A_2\|^2}{2N_2 \cdot A_2A_1}. \quad (1)$$

Then,

$$\Omega = \frac{\Omega_1 + \Omega_2}{2} + \frac{(\lambda_1^2 - \lambda_2^2)}{2} \frac{\Omega_1 \Omega_2}{\|\Omega_1 \Omega_2\|^2}. \quad (2)$$

The radius $R$ of the circle satisfies

$$4R^2 = 4(\lambda_1^2 + \lambda_2^2) - \|\Omega_1 \Omega_2\|^2 - \frac{(\lambda_1^2 - \lambda_2^2)^2}{\|\Omega_1 \Omega_2\|^2}. \quad (3)$$
We now express the previous formulae with the coordinates of

\[ A_1 = (0, 0, 0), \quad N_1 = (0, 0, 1), \quad N_2 = (l, m, n), \quad A_2 = (0, y_2, z_2). \]

Lemma:

\[
S = 4 \frac{\| \Omega_1 \Omega_2 \|^2}{\| A_1 A_2 \|^2} \cdot (\overrightarrow{N_1} \cdot \overrightarrow{A_1 A_2}) \cdot (\overrightarrow{N_2} \cdot \overrightarrow{A_2 A_1})
\]

is a polynomial in the input parameters.

Indeed,

\[
S = l^2 (y_2^2 z_2^2 + z_2^4) + m^2 (y_2^2 - z_2^2)^2 + 4 mn y_2 z_2 (y_2^2 - z_2^2) + 4 n^2 y_2^2 z_2^2.
\]
In addition, the following properties holds:

- \( 4 \cdot S \cdot R^2 = \|A_1A_2\|^2(S + I^2y_2^2(y_2^2 + z_2^2)), \) is a polynomial.

- The coordinates of \( S \cdot \Omega \bar{M} \) are polynomials in the input parameters.

- The quantity \( (\Omega_1\Omega_2 \cdot \Omega \bar{M}) \cdot (N_1 \cdot \bar{A}_1 \bar{A}_2) \cdot (N_2 \cdot \bar{A}_2 \bar{A}_1) \) is a polynomial in the input parameters, namely

\[
X (ly_2^2z_2 + lz_2^3) + Y (-my_2^2z_2 + mz_2^3 - 2ny_2z_2^2) + \\
+ Z (my_2^3 - my_2z_2^2 + 2ny_2^2z_2).
\]

It provides a reduced equation of the supporting plane of the circle.
The squared distance of a point $M$ to the skeletal circle is
\[
(\vec{N} \cdot \vec{\Omega M})^2 + \left( \sqrt{\|\Omega M\|^2 - (\vec{N} \cdot \vec{\Omega M})^2} - R \right)^2.
\]
hence, the equation is:
\[
\left(\|\Omega M\|^2 + R^2 - r^2\right)^2 - 4R^2 \left(\|\Omega M\|^2 - (\vec{N} \cdot \vec{\Omega M})^2\right) = 0.
\]
Finally, we drop the hypothesis that $\vec{N}$ is a unitary vector:
\[
\|N\|^2 \left(\|\Omega M\|^2 + R^2 - r^2\right)^2 - 4R^2 \left(\|N\|^2 \cdot \|\Omega M\|^2 - (\vec{N} \cdot \vec{\Omega M})^2\right) = 0.
\]
Proof of our theorem

Let $P_1$ be at the origin, $N_1 = (0, 0, 1)$, $P_2 = (0, y_2, z_2)$; $N_2 = (l, m, n)$ and $P_3 = (x_3, y_3, z_3)$.

Introduce $r_1, r_2, A_1 = P_1 + r_1 N_1$, $A_2 = P_2 + r_2 N_2$, with

$$r_1^2 = r_2^2 \|N_2\|^2 = r_2^2(l^2 + m^2 + n^2). \quad (4)$$

Then the following **polynomial** quantities:

$$P := \|A_1 A_2\|^2 = r_2^2 l^2 + m^2 r_2^2 + n^2 r_2^2 + 2m r_2 y_2 - 2n r_1 r_2 - 2n r_2 z_2 + r_1^2 - 2r_1 z_2 + y_2^2 + z_2^2,$$

$$Q_1 := N_1 \cdot A_1 A_2 = n r_2 - r_1 + z_2,$$

$$Q_2 := N_2 \cdot A_2 A_1 = l^2 r_2 + m^2 r_2 + n^2 r_2 + m y_2 - n r_1 + n z_2.$$
Using the properties given in our Lemma, the implicit equation of the torus writes:

\[
\frac{P}{Q_1^2 \cdot Q_2^2} \cdot E(x, y, z, x_2, y_2, l, m, n, r_1, r_2)
\]

where \(E\) is a **polynomial**.

The vanishing of \(P, Q_1, Q_2\) is not relevant to our problem.

Symbolic computations show that \(E\) is a polynomial
- of degree 4 in the variables \(x, y, z\);
- of degree 8 in the variables \(l, m\);
- of degree 6 in the variables \(n, y_2, z_2, r_2\);
- of degree 7 in the variable \(r_1\).
So, we got two algebraic equations
\[ E(x_3, y_3, z_3, x_2, y_2, l, m, n, r_1, r_2) = 0, \]
\[ U := r_1^2 - r_2^2(l^2 + m^2 + n^2) = 0 \]

We divide \( E \) by \( U \) w.r.t. \( r_1 \) and get \( S_1 + r_1 S_2 \).

So, we have the system

\[ S_1 + r_1 S_2 = 0, \quad r_1^2 - r_2^2(l^2 + m^2 + n^2). \] (6)

where \( S_1 \) and \( S_2 \) are polynomials of degree 4 and 3.

The elimination of \( r_1 \) yields a polynomial
in \( x_3, y_3, z_3, l, m, n, y_2, z_2, r_2 \)
of degree 8 w.r.t. the variable \( r_2 \). Then \( r_1 = -\frac{S_1 S_2}{S_2} \).
A maximal configuration

**Figure:** Eight smooth tori that interpolate a point set.

*Busé-Galligo, MEGA17, Nice*  
Extraction of tori from minimal point sets
Proportion of the number of smooth tori found through ten thousands random point sets.

<table>
<thead>
<tr>
<th>Number of tori</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Proportion (%)</td>
<td>7.98</td>
<td>39.42</td>
<td>29.71</td>
<td>16.61</td>
<td>5.01</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Number of tori</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>Proportion (%)</td>
<td>0.77</td>
<td>0.47</td>
<td>0.02</td>
<td>0.01</td>
</tr>
</tbody>
</table>

The average time was **24 ms** (almost independently of the point set).
For the applications, we also require that $R > r$. 
Conclusion

- We studied efficient fitting algorithms to find tori passing through minimal sets of points or points with normals.
- Our analysis reduced to quantify the study a 3D circle interpolation problem.
- We got a fast algorithm, with simple techniques inspired by classical effective algebraic geometry.
- Its main step consists of evaluating polynomials and applying rather simple matrix-based closed formulae.
- The resulting procedure is much more efficient than the existing extraction procedures; it will be implemented in C++ and included in the specialized softwares.
Thank you for your attention