Computing the monodromy and pole order filtration on Milnor fiber cohomology of plane curves

Alexandru Dimca [1] Gabriel Sticlaru [2]

[1] Université Côte d'Azur, LJAD

[2] Ovidius University

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http://math1.unice.fr/ dimca/singular.html

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Basic definitions

Let $C : f(x, y, z) = 0$ be a reduced plane curve in the complex projective plane $\mathbb{P}^2$, defined by a degree d homogeneous polynomial f in the graded polynomial ring $S = \mathbb{C}[x, y, z]$.

The smooth affine surface $F : f(x, y, z) = 1$ in $\mathbb{C}^3$ is called the Milnor fiber of f. The mapping $h : F \to F$ given by $(x, y, z) \mapsto (\theta x, \theta y, \theta z)$ for $\theta = \exp(2\pi i/d)$ is called the monodromy of f.

There are induced monodromy operators $h^j : H^j(F, \mathbb{Q}) \to H^j(F, \mathbb{Q})$, $h^j(\omega) = (h^{-1})^*(\omega)$, for $j = 0, 1, 2$, and we can look at the corresponding characteristic polynomials

$$\Delta^j_C(t) = \det(t \cdot Id - h^j | H^j(F, \mathbb{Q})).$$

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$$\Delta_C^j(t) = \det(t \cdot Id - h^j | H^j(F,\mathbb{Q})).$$

Basic facts: Alexander polynomial

Proposition

1. *One has $h^d = Id$, and hence the monodromy operators h^j are semisimple.*
2. *The roots of the characteristic polynomials $\Delta^j_C(t)$ are d-th roots of unity.*
3. $H^0(F, \mathbb{Q}) = \mathbb{Q}$ *and* $h^0 = Id$.
4. $$\Delta^0_C(t)\Delta^1_C(t)^{-1}\Delta^2_C(t) = (t^d - 1)^{\chi(U)},$$ *where $\chi(U)$ denotes the topological Euler characteristic of the complement $U = \mathbb{P}^2 \setminus C$.*

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Basic facts:Hodge theory

Theorem (A.D., G. Lehrer, S. Papadima, M. Saito)

There is a direct sum decomposition

$$H^1(F,\mathbb{Q}) = H^1(F,\mathbb{Q})_1 \oplus H^1(F,\mathbb{Q})_{\neq 1}$$

according to eigenspaces of the monodromy operator h^1 *such that*

1. $H^1(F,\mathbb{Q})_1 = H^1(U,\mathbb{Q})$ *is a pure Hodge structure of type* $(1,1)$*, of dimension* $r-1$*, where* r *is the number of irreducible components of the curve* C*;*
2. $H^1(F,\mathbb{Q})_{\neq 1}$ *is a pure Hodge structure of weight 1.*

In particular, for $\lambda \neq 1$ an eigenvalue of h^1, we have

$$H^1(F,\mathbb{C})_\lambda = H^{1,0}(F,\mathbb{C})_\lambda \oplus H^{0,1}(F,\mathbb{C})_\lambda$$

and $\dim H^{1,0}(F,\mathbb{C})_\lambda = \dim H^{0,1}(F,\mathbb{C})_{\overline{\lambda}}$.

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Main question

Question

Given the degree d reduced plane curve $C : f = 0$ and a d-th root of unity $\lambda \neq 1$, determine the multiplicity $m(\lambda)$ of λ as a root of the Alexander polynomial $\Delta_C(t)$.

A more precise question: find a basis (e.g. in terms of rational differential forms) for the eigenspace $H^1(F, \mathbb{C})_\lambda$ or for $H^{1,0}(F, \mathbb{C})_\lambda$.

The complete answer is known in the case C smooth and for $H^2(F, \mathbb{C})_\lambda$ (in fact for any smooth projective hypersurface) by the work of Ph. Griffiths (1969) and J. Steenbrink (1977).

For singular curves, several answers have been given by H. Esnault (1982), A. Libgober (1982), E. Artal-Bartolo (1990) and many others.

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A general spectral sequence

Let Ω^j denote the graded S-module of (polynomial) differential j-forms on $\mathbb{C}^3$, for $0 \leq j \leq 3$. For instance $f \in \Omega^0 = S$ and

$$df = f_x dx + f_y dy + f_z dz \in \Omega^1.$$

The complex $K_f^* = (\Omega^*, df\wedge)$ is nothing else but the Koszul complex in S of the partial derivatives f_x, f_y and f_z of the polynomial f. Fix an integer k such that $1 \leq k \leq d$ and set from now on

$$\lambda = \exp(-2\pi ik/d).$$

Then one has the following results.

Theorem (A.D. (1990), A.D. and M. Saito (2012), M. Saito (2016))

With the above notation, for any integer k with $1 \leq k \leq d$, there is an E_1 - spectral sequence $E_*(f)_k$ such that

$$E_1^{s,t}(f)_k = H^{s+t+1}(K_f^*)_{td+k}$$

and converging to

$$E_\infty^{s,t}(f)_k = Gr_P^s H^{s+t}(F, \mathbb{C})_\lambda$$

where P^* is a decreasing filtration on the Milnor fiber cohomology, called the pole order filtration. Moreover

- *$E_1(f)_k = E_\infty(f)_k$ for all k's if and only if C is smooth.*
- *$E_2(f)_k = E_\infty(f)_k$ for all k's if and only if C has only weighted homogeneous singularities.*

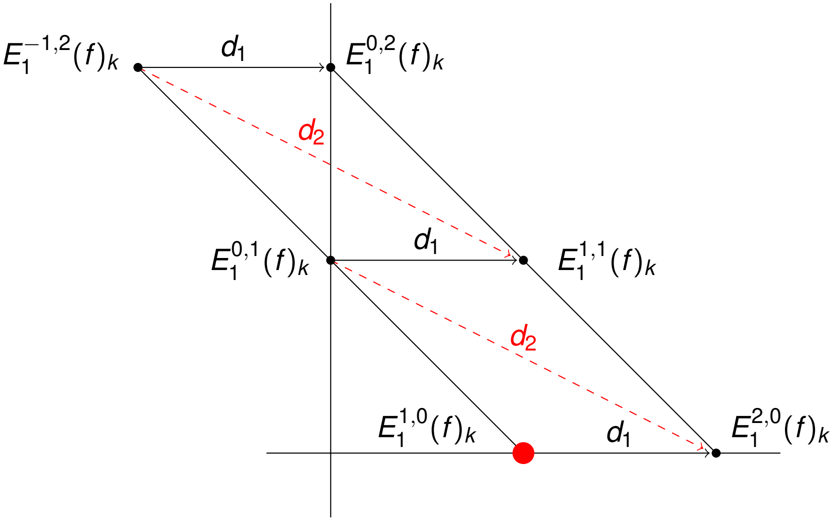

Figure: The E_1-term of the spectral sequence $E_*(f)_k$, $d_1([\omega]) = [d(\omega)]$.

Back to Alexander polynomials

What about the curves with non weighted homogeneous singularities? Then it is known that $E_2(f)_k \neq E_\infty(f)_k$ for any $1 \leq k \leq d$. However we have the following.

Theorem

Let $C : f = 0$ be a reduced degree d curve , and let $\lambda = \exp(-2\pi ik/d)$, with $k \in (0, d)$ an integer. Then λ is a root of the Alexander polynomial $\Delta_C(t)$ of multiplicity $m(\lambda)$ given by

$$m(\lambda) = \dim E_2^{1,0}(f)_k + \dim E_2^{1,0}(f)_{k'}.$$

Main idea: show that the pole order filtration on $H^1(F, \mathbb{C})$ coincides with the Hodge filtration.

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The algorithm

The following steps can be performed using the software SINGULAR for instance.

1. Define $Syz(f) := \ker\{df\wedge : \Omega^2 \to \Omega^3\}$, and determine a $\mathbb{C}$-vector basis of the q-th homogeneous component $Syz(f)_q$, for $2 \leq q \leq 2d$. This is particularly easy when C is a free curve.
2. Determine the dimension κ_q of the kernel of the morphism $\delta_q : Syz(f)_q \to M(f)_{q-3}$, where $M(f) = S/(f_x, f_y, f_z)$ is the Milnor algebra of f and

   $$\delta_q(ady \wedge dz - bdx \wedge dz + cdx \wedge dy) = [a_x + b_y + c_z] \in M(f)_{q-3}.$$

   δ_q is the differential d_1 in the spectral sequence.
3. Compute $\epsilon_q = \kappa_q - \dim(df \wedge \Omega^1)_q$, where the last dimension depends only on the degree d, being exactly $3 \dim S_{q-d-1}$.

Finally note that

$$\epsilon_q = \dim E_2^{1-t,t}(f)_k$$

where $k \in [1, d]$, $q - k$ is divisible by d and $t := (q - k)/d$. This gives us all the E_2-terms of the spectral sequence that occur in our main results above.

Moreover, let $\iota : F \to \mathbb{C}^3$ denote the inclusion of the Milnor fiber F into $\mathbb{C}^3$ and $\Delta : \Omega^j \to \Omega^{j-1}$ denote the contraction with the Euler vector field $E = x\partial_x + y\partial_y + z\partial_z$. If a 2-form η satisfies both

$$df \wedge \eta = 0 \text{ and } d\eta = 0,$$

then it gives rise to an element $\alpha = \iota^*(\Delta(\eta))$ in $H^1(F, \mathbb{C})$. In this way, the above algorithm can give a basis for the cohomology group $H^1(F, \mathbb{C})$.

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Zariski's sextic with 6 cusps on a conic

Consider the sextic curve

$$C : f = (x^2 + y^2)^3 + (y^3 + z^3)^2 = 0,$$

having 6 cusps on a conic and Alexander polynomial $\Delta_C(t) = t^2 - t + 1$. This curve is not free, the module $Syz(f)$ has a minimal set of generators consisting of one syzygy of degree 5, namely

$$\omega_1 = yz^2 dy \wedge dz + xz^2 dx \wedge dz + xy^2 dx \wedge dy,$$

and 3 other syzygies of degree 7, among which

$$\omega_2 = (y^3 z^2 + z^5) dy \wedge dz - (x^5 + 2x^3 y^2 + xy^4) dx \wedge dy.$$

It is clear that $d(\omega_1) = d(\omega_2) = 0$.

Theorem

Let $C : f = (x^2 + y^2)^3 + (y^3 + z^3)^2 = 0$ *be the above sextic curve and set* $\lambda = \exp(\pi i/3)$. *Then the following hold.*

1. $\alpha = \iota^*(\Delta(\omega_1))$ *spans the 1-dimensional vector space* $H^{1,0}(F, \mathbb{C})_\lambda = H^1(F, \mathbb{C})_\lambda$.
2. $\beta = \iota^*(\Delta(\omega_2))$ *spans the 1-dimensional vector space* $H^{0,1}(F, \mathbb{C})_{\overline{\lambda}} = H^1(F, \mathbb{C})_{\overline{\lambda}}$.

Hence the pair α, β gives a basis for the cohomology group

$$H^1(F, \mathbb{C}) = H^1(F, \mathbb{C})_\lambda \oplus H^1(F, \mathbb{C})_{\overline{\lambda}}.$$

A SINGULAR output for Zariski sextic

For the he term $E_2 = E_3 = E_\infty$, we set $\epsilon^2_q = \dim E_2^{1-t,t}(f)_k$ and $\mu^2_q = \dim E_2^{2-t,t}(f)_k$, where $q = td + k$.

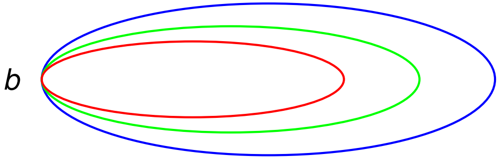
q:	3	4	5	6	7	8	9	10	11	12
ϵ^2_q:	0	0	1	0	1	0	0	0	0	0
μ^2_q:	1	3	6	7	10	9	8	6	4	1

The sequence $\epsilon^2_q = \epsilon^\infty_q$ is symmetric with respect to the center $q = d$, while the sequence $\mu^2_q = \mu^\infty_q$ does not have this property.

A free curve with non-w. h. singularities

Consider the following conic pencil with one point base locus:

$$tx^2 + s(xz + y^2) = 0$$



Using this pencil we construct the following curves:

$C_{2m} : f = x^{2m} + (xz + y^2)^m = 0$ and $C_{2m+1} : f = x(x^{2m} + (xz + y^2)^m) = 0.$

These curves have been essentially introduced by C.T.C. Wall and independently by Arkadiusz Płoski.

Theorem (A.D. 2017)

Consider the curves C_d defined above, for $d \geq 3$. Then the following holds.

1. *The curves C_d are free with exponents $d_1 = 1$ and $d_2 = d - 2$.*
2. *The complement U satisfies $b_2(U) = 0$ and the Euler characteristic $\chi(U)$ is given by*

$$\chi(U) = 2 - d + \lfloor \frac{d}{2} \rfloor.$$

3. *When d is odd, then the Milnor fiber F is homotopy equivalent to a bouquet of circles $\vee S^1$, and hence the corresponding Alexander polynomial $\Delta(t)$ of C_d is given by*

$$\Delta(t) = (t-1)(t^d - 1)^{-\chi(U)}.$$

A SINGULAR output for $d = 9$

The term E_2

q :	3	4	5	6	7	8	9	10	11	12	13	14	15
ϵ_q^2 :	1	1	2	2	3	3	4	3	3	3	3	3	3
μ_q^2 :	1	1	2	2	3	3	3	3	3	3	3	3	3

The term $E_3 = E_\infty$

q :	3	4	5	6	7	8	9	10	11	12	13	14	15
ϵ_q^∞ :	1	1	2	2	3	3	4	3	3	2	2	1	1
μ_q^∞ :	0	0	0	0	0	0	0	0	0	0	0	0	0

Conjecture (A.D. and G.S. 2016)

With the above notation, for any reduced degree d plane curve $C : f = 0$ and any integer k with $1 \leq k \leq d$, the E_1- spectral sequence $E_*(f)_k$ degenerates at the third page, i.e.

$$E_3(f)_k = E_\infty(f)_k.$$

Note that there are similar spectral sequences for a reduced degree d hypersurface $V : f = 0$ in $\mathbb{P}^n$, for $n > 2$, whose complexity increases according to the dimension of the singular set of V.

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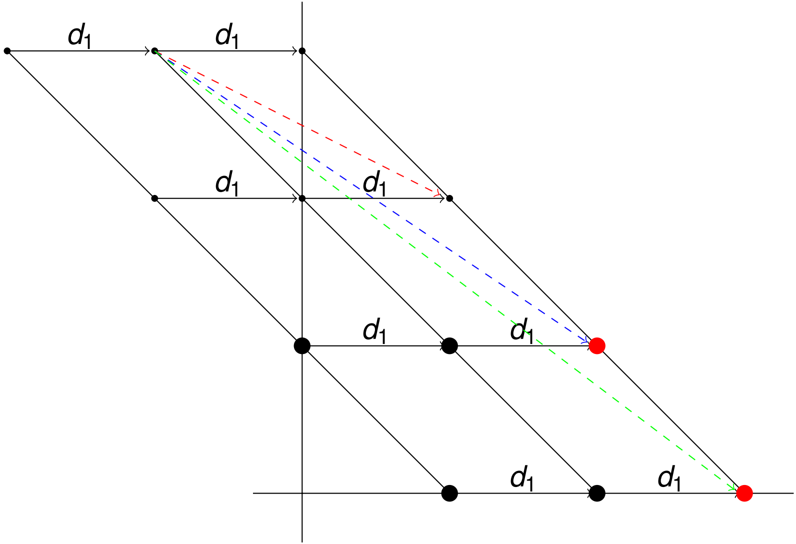

Figure: The spectral sequence $E_*(f)_k$ for a surface with 1-dim singular set.

Conjecture (A.D. and G.S. 2017)

For any hyperplane arrangement $V : f = 0$ in $\mathbb{P}^n$ and any integer k with $1 \leq k \leq d$, d being the number of hyperplanes in V, the E_1- spectral sequence $E_*(f)_k$ degenerates at the second page, i.e.

$$E_2(f)_k = E_\infty(f)_k.$$

Moreover, one has $E_2^{s,t}(f)_k = 0$ for $t > 1$.

One knows that $E_\infty^{s,t}(f)_k = 0$ for $t > 1$ using results by M. Saito on the Bernstein-Sato polynomials of hyperplane arrangements. A similar result holds for free locally quasi-homogeneous hypersurfaces $V : f = 0$ using results by L. Narvéz Macarro.

Definition

Let $V : f = 0$ be a hypersurface of degree d in $\mathbb{P}^n$ and let k be a positive integer satisfying $1 \leq k \leq d$. We say that the hypersurface V, or the defining polynomial f, is k-top-computable if

$$\dim E_2^{n,0}(f)_k + \dim E_2^{n-1,1}(f)_k = \dim H^n(F, \mathbb{C})_\lambda,$$

where $\lambda = \exp(-2\pi i k/d) \neq 1$, and respectively
$\dim E_2^{n,0}(f)_d = \dim H^n(F, \mathbb{C})_1$ if $k = d$.

Conjecture

For any arrangement $\mathcal{A} : f = 0$ of d hyperplanes in $\mathbb{P}^n$, and for any free locally quasi-homogeneous divisor $V : f = 0$ of degree d in $\mathbb{P}^n$, the defining polynomial f is k-top-computable for any positive integer k satisfying $1 \leq k \leq d$.

An Example

Example (The Coxeter arrangement D_4)

The arrangement D_4 is defined in $\mathbb{C}^4$ by the equation of degree 12

$$\mathcal{A} : f = (x^2 - y^2)(x^2 - z^2)(x^2 - w^2)(y^2 - z^2)(y^2 - w^2)(z^2 - w^2) = 0.$$

We have the following formulas for the Alexander polynomials:

$$\Delta^1(\mathcal{A}) = \Phi_1^{11}\Phi_3,$$

$$\Delta^2(\mathcal{A}) = \Phi_1^{39} \cdot \Phi_2^4 \cdot \Phi_3^9 \cdot \Phi_6^4,$$

and

$$\Delta^3(\mathcal{A}) = \Phi_1^{45} \cdot \Phi_2^{20} \cdot \Phi_3^{24} \cdot \Phi_4^{16} \cdot \Phi_6^{20} \cdot \Phi_{12}^{16},$$

where Φ_j is the j-th cyclotomic polynomial. Any value $k \neq 2, 4, 6, 12$ is non-resonant with respect to the arrangement D_4.