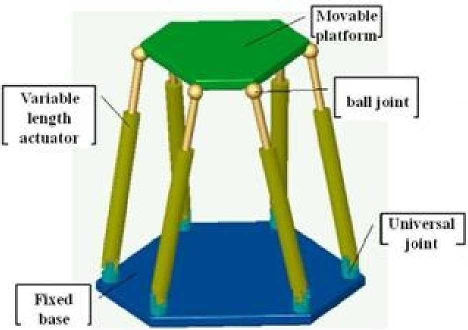
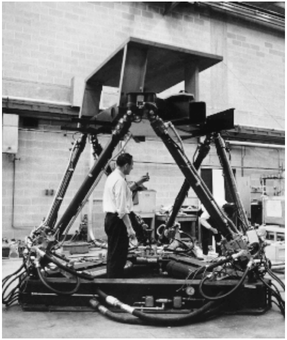
Rationality of the set of singular poses of a Gough-Stewart platform

Michel Coste – IRMAR (Rennes, FR)
joint work with Seydou Moussa, U. of Maradi, Niger

Workshop on
Singularities in Real and Complex Geometry
Nice, June 16, 2017

Gough-Stewart platform





- Each leg has an actuated prismatic joint. The universal (Cardan) and ball joints are passive.
- 6 degrees of freedom

Workspace and joint space
Inverse Kinematics and Direct Kinematics

- *Workspace*: the group of rigid motions $\mathrm{SE}(3) \simeq \mathbb{R}^3 \rtimes \mathrm{SO}(3)$, $(\vec{t}, R) : \vec{x} \mapsto R\vec{x} + \vec{t}$, 6-dimensional.
- (actuated) *Joint space*: $(\mathbb{R}_+)^6$, coordinates $=$ lengths of legs.
- *Inverse Kinematics Mapping* (IKM):

$$\begin{aligned} \text{Workspace} &\longrightarrow \text{Joint space} \\ (\vec{t}, R) &\longmapsto (r_1, \ldots, r_6) \end{aligned}$$

- *Direct Kinematics Problem*: given the lengths of legs, how many assembly modes (poses of the platform where the legs have the given lengths) ? 40 complex solutions (Lazard, Mourrain, Ronga & Vust), all of which can be real (Dietmaier).

Singularities and Jacobian of the IKM

- Is it possible to follow continuously the solutions of the DKP (assembly modes) when we change the lengths of legs ?
- Local inversion theorem : yes, if the Jacobian determinant of the IKM does not vanish.
- If not, parallel singularity (singularity of the IKM, its differential is not invertible). Loss of control of a degree of freedom, possible destruction of the platform.
- Differential of the IKM :

$$\begin{aligned} \mathfrak{se}(3) &\longrightarrow \mathbb{R}^6 \\ \vec{\mathcal{T}}_{\mathrm{platform}} &\longmapsto (\dot{r}_1, \ldots, \dot{r}_6) \end{aligned}$$

$\vec{\mathcal{T}}_{\mathrm{platform}}$: field of velocities of the platform (twist),
$\dot{r}_i$: joint velocities.

Screws, twists and wrenches

- Screw: vector field $\vec{\mathcal{T}}$ on euclidean 3-space of the form $\vec{\mathcal{T}}(M) = \vec{q} + \vec{p} \times \overrightarrow{OM}$. The reduct at O of the screw is $(\vec{p}, \vec{q})$.
- Twist: field of velocities of a rigid body, $\vec{v}_M = \vec{v}_O + \vec{\omega} \times \overrightarrow{OM}$, where $\vec{\omega}$ is the angular velocity vector. Reduct at O: $(\vec{\omega}, \vec{v}_O)$. The twists form a 6-dimensional space, the Lie algebra $\mathfrak{se}(3)$. Matrix form: $\begin{pmatrix} A_{\vec{\omega}} & \vec{v}_O \\ 0 & 0 \end{pmatrix}$, where $A_{\vec{\omega}}$ is the skew-symmetric matrix of $\vec{\omega} \times \cdot$.
- Wrench: field of torques (moments) of a system of forces, $\vec{m}_M = \vec{m}_O + \vec{f} \times \overrightarrow{OM}$, where $\vec{f}$ is the resultant of the system of forces. Reduct at O : $(\vec{f}, \vec{m}_O)$.

Reciprocal product. Plücker coordinates

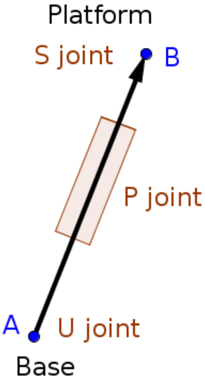
- *Reciprocal product* of two screws $\vec{\mathcal{T}}_1$ and $\vec{\mathcal{T}}_2$ with reducts $(\vec{p}_1, \vec{q}_1)$ and $(\vec{p}_2, \vec{q}_2)$: $\vec{\mathcal{T}}_1 \odot \vec{\mathcal{T}}_2 = \vec{p}_1 \cdot \vec{q}_2 + \vec{q}_1 \cdot \vec{p}_2$. Independent of the reduction point, nondegenerate symmetric bilinear form.
- Reciprocal product of a wrench and a twist = power produced.
- Reciprocal screws: orthogonal w.r.t. reciprocal product.
- *Plücker coordinates* of a line: reduct of a pure force along this line (homogeneous coordinates). For a line (AB): $\mathcal{P}_{A,B} = (\vec{AB}, \vec{OA} \times \vec{AB})$. Plücker coordinates for the line at infinity of a plane orthogonal to $\vec{n}$: $(\vec{0}, \vec{n})$.
- Plücker coordinates of lines are the nonzero self-reciprocal screws. Lines are intersecting (or parallel) iff their Plücker coordinates are reciprocal.

Screw analysis of the singularities

- Decomposition of the twist of the platform along a leg $A_i B_i$, according to joints (composition of velocities): $\vec{\mathcal{T}}_{\text{platform}} = \vec{\mathcal{T}}_{U_i} + \vec{\mathcal{T}}_{P_i} + \vec{\mathcal{T}}_{S_i}$.
- Reciprocal product with $\mathcal{P}_{A_i,B_i}$: no power produced at U_i and S_i, so

$$\mathcal{P}_{A_i,B_i} \odot \vec{\mathcal{T}}_{\text{platform}} = \mathcal{P}_{A_i,B_i} \odot \vec{\mathcal{T}}_{P_i} = r_i \, \dot{r}_i \ .$$

- A pose is singular iff the Plücker coordinates of the legs are linearly dependent.



Singularity hypersurface

- Set of singular poses: hypersurface $\mathrm{Sing} \subset \mathrm{SE}(3)$ with equation the determinant of the 6×6 matrix with rows

$$(R\vec{b}_i + \vec{t} - \vec{a}_i\,,\ \vec{a}_i \times (R\vec{b}_i + \vec{t} - \vec{a}_i))$$

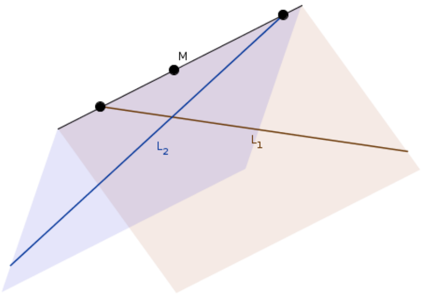
  ($\vec{a}_i$ and $\vec{b}_i$ coordinates of A_i and B_i in frames attached to the base and to the platform resp.)
- Degree 3 equation w.r.t. $\vec{t} \in \mathbb{R}^3$ (use generalized Laplace expansion along first 3 columns). Fixing R (orientation of platform), cubic surface $\mathrm{Sing}_R \subset \mathbb{R}^3$.
- Sing is a family of affine cubic surfaces parametrized by SO_3.

Rational parametrization ?

A complete description and characterization of the singularities would be to parametrize the entire singularity hypersurface in the task-space (6D in the case of the Stewart platform).

- Is Sing a rational variety ?
- SO_3 is rational (parametrization of rotation matrices by quaternions). The field $K = \mathbb{R}(\mathrm{SO}_3)$ is a purely transcendental extension of $\mathbb{R}$.
- $(\mathrm{Sing}_R)_{R \in \mathrm{SO}_3}$ defines an affine cubic surface $\Sigma \subset K^3$ over K (with the same equation as Sing, but now entries of R in the base field K). Rationality of Sing over $\mathbb{R}$ reduces to the rationality of Σ over K. Rationality of cubic surfaces is a classical problem in algebraic geometry.

Rational parametrization of a cubic surface using two skew lines



- L_1, L_2 two skew lines on a projective nonsingular cubic surface C.

  $$M \mapsto ((L_1, M), (L_2, M))$$
  $$C \to \mathbf{P}^1 \times \mathbf{P}^1$$

  is a birational regular mapping: a blowing-up in 5 points.

- Does kinematics give a birational regular mapping from Σ to a quadric surface defined over $\mathbb{R}(\mathrm{SO}_3)$?

Reciprocal screws again

- Singular pose: there is a nonzero screw reciprocal to the systems of Plücker coordinates of all 6 legs. For a general Gough Stewart platform (base and platform are not assumed planar), Σ is non singular. Hence, for every point of Σ there is a *line* of reciprocal screws.
- We obtain a regular morphism $\Sigma \to \mathbf{P}^5$ defined over $K = \mathbb{R}(\mathrm{SO}_3)$. This is a birational morphism to a nonsingular quadric surface $Q \subset \mathbf{P}^5$ defined over K, with a K-rational point (image of a K-rational point of Σ). This shows:

Theorem

Σ is rational over K. Hence, $\mathrm{Sing} \subset \mathrm{SE}(3)$ is rational over $\mathbb{R}$.

Computations

- Reciprocity equations: homogeneous linear system in $\vec{p}, \vec{q}$, with parameter $\vec{t}$. Can assume $\vec{a}_1 = \vec{b}_1 = 0$. First eqn $\vec{t} \cdot \vec{q} = 0$, other eqns

$$\vec{c}_i \cdot \vec{q} + (\vec{a}_i \times (\vec{c}_i + \vec{t})) \cdot \vec{p} = 0 \ ,$$

  where $\vec{c}_i = R\vec{b}_i - \vec{a}_i$. $\vec{a}_i$ and $\vec{c}_i$ have coefficients in the base field K.
- Eliminating $\vec{t}$, 1st step: writing $\vec{a}_5$ and $\vec{a}_6$ as linear combinations of $\vec{a}_2, \vec{a}_3, \vec{a}_4$ in eqns 5,6 gives *2 linear homogeneous equations in $\vec{p}$, $\vec{q}$*, without $\vec{t}$.
- Eliminating $\vec{t}$, second step: the identity

$$[\vec{p}\cdot(\vec{a}_3\times\vec{a}_4)]\,\vec{p}\times\vec{a}_2+[\vec{p}\cdot(\vec{a}_4\times\vec{a}_2)]\,\vec{p}\times\vec{a}_3+[\vec{p}\cdot(\vec{a}_2\times\vec{a}_3)]\,\vec{p}\times\vec{a}_4 = 0$$

  allows to get from eqns 2,3,4 a *homogeneous quadratic equation in $\vec{p}, \vec{q}$*, without $\vec{t}$.

So far, so good, but...

there are still many things to understand. For instance: does the regular mapping $\Sigma \to Q$ extend to the projective closure Σ^h of Σ (which is also nonsingular)?

Question: Let $A(\mathbf{x})$ be a $n \times n$ matrix whose entries are affine functions of $\mathbf{x} = (x_1, \ldots, x_p)$. Assume that the projective closure Σ^h of $\Sigma = \{\mathbf{x} \mid \det(A(\mathbf{x})) = 0\}$ is a nonsingular hypersurface in $\mathbb{P}^p$. Does the regular mapping $\Sigma \to \mathbb{P}^{n-1}$ sending $\mathbf{x}$ to $\ker(A(\mathbf{x}))$ extend to the projective closure Σ^h? This is not clear (to me) when $\deg(\det(A(\mathbf{x}))) < n$, i.e. when the homogenization of the determinant is not the determinant of the matrix with homogenized entries.

Kinematics of exceptional divisors

- The (supposed) birational regular morphism $\Sigma^h \to Q$ is a succession of blowing-up. It has to be the blowing-up of the quadric Q in 5 points.
- There are exactly two lines in Σ^h intersecting all 5 exceptional divisors. Computations of examples verify that these correspond to the degree 2 factor of the polynomial giving the 27 lines (other factors of degrees 5,10,10).
- Kinematics significance of these 5 exceptional divisors: there is a line intersecting all legs