Bar Code for monomial ideals

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Introduction: strongly stable

Strongly stable ideals play a special role in the study of Hilbert scheme:

- Bayer and Eisenbud
- Galligo
- Eisenbud and Peeva
- Aramova-Herzog

Introduction: stable

The notion of *stable ideal* has been introduced as a generalization of 0-Borel-fixed ideals.

- Eliahou-Kervaire
- Bigatti and Hulett

Much research…

In connection with the study of Hilbert schemes it has been considered relevant to list all the stable ideals and strongly stable ideals with a fixed Hilbert polynomial.

- Bertone - Cioffi - Lella - Roggero
- Nagel - Moore
- ...

What about counting them?

Stable VS Strongly stable

A monomial ideal $J \triangleleft \mathcal{P} = \mathbf{k}[x_1, ..., x_n]$ is called **stable** if it holds

$$\tau \in J,\ x_j > \min(\tau) \Longrightarrow \frac{x_j \tau}{\min(\tau)} \in J.$$

A monomial ideal $I \triangleleft \mathcal{P} = \mathbf{k}[x_1, ..., x_n]$ is called **strongly stable** if, for every term $\tau \in I$ and pair of variables $x_i,\ x_j$ such that $x_i | \tau$ and $x_i < x_j$, then also $\frac{\tau x_j}{x_i}$ belongs to I or, equivalently, for every $\sigma \in \mathrm{N}(I)$, and pair of variables $x_i,\ x_j$ such that $x_i | \sigma$ and $x_i > x_j$, then also $\frac{\sigma x_j}{x_i}$ belongs to $\mathrm{N}(I)$.

Example

In $k[x_1, x_2, x_3]$ with $x_1 < x_2 < x_3$:

- the ideal $I_1 = (x_1^3, x_1x_2, x_2^2, x_1^2x_3, x_2x_3, x_3^2)$ is **stable**. Anyway, it is **not strongly stable**, since $x_1x_2 \in I_1$, but $\frac{(x_1x_2)x_3}{x_2} = x_1x_3 \notin I_1$;

- the ideal $I_2 = (x_1^2, x_1x_2, x_2^2, x_3)$ is **strongly stable**.

But, in a strange case....

The case of two variables is very simple:

Lemma

An ideal $I \triangleleft k[x_1, x_2]$ is stable if and only if it is strongly stable.

Counting in two variables? And three?

And more?

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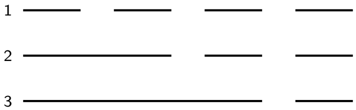
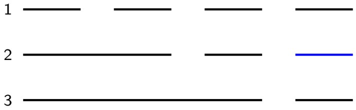
Bar Code

A Bar Code B is a picture composed by segments, called **bars**, superimposed in horizontal rows



Bar Code

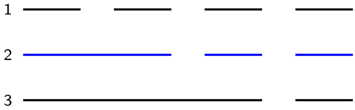
$\mathrm{B}_j^{(i)}$ is the j-th bar (from left to right) of the i-th row (from top to bottom), i.e. the ***j*-th *i*-bar**



$B_3^{(2)}$= third bar in the second row.

Bar Code

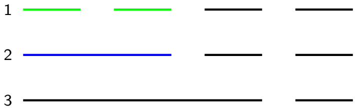
$\mu(i)$ is the number of bars of the i-th row.
Bar list :$L_B := (\mu(1), ..., \mu(n))$.



E.g. $\mu(2) = 3$, so there are three bars in the second row.
$L_B := (4, 3, 2)$.

Bar Code

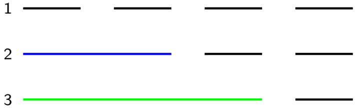
$l_1(\mathrm{B}_j^{(1)}) := 1$, $\forall j \in \{1, 2, ..., \mu(1)\}$ is the $(1-)$*length* of the 1-bars; $l_i(\mathrm{B}_j^{(k)})$, $2 \leq k \leq n$, $1 \leq i \leq k-1$, $1 \leq j \leq \mu(k)$ the *i-length* of $\mathrm{B}_j^{(k)}$, i.e. the number of *i*-bars lying over $\mathrm{B}_j^{(k)}$



$l_1(B_1^{(1)}) = 1$; $l_1(B_1^{(2)}) = 2$.

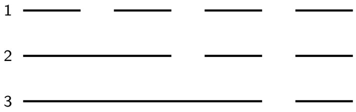
Bar Code

$\forall i, j,\ 1 \leq i \leq n-1,\ 1 \leq j \leq \mu(i),\ \exists! \bar{j} \in \{1, ..., \mu(i+1)\}$ s.t. $\mathsf{B}_{\bar{j}}^{(i+1)}$ lies under $\mathsf{B}_{j}^{(i)}$



Bar Code

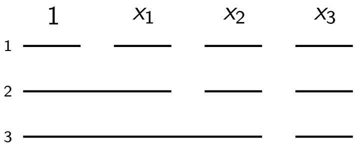
$$\forall i_1,\ i_2 \in \{1, ..., n\},\ \sum_{j_1=1}^{\mu(i_1)} l_1(\mathsf{B}_{j_1}^{(i_1)}) = \sum_{j_2=1}^{\mu(i_2)} l_1(\mathsf{B}_{j_2}^{(i_2)})$$



All the bars have the same 1-length, that is, 4.

Bar codes and order ideals

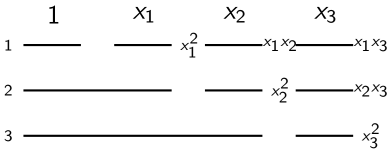
Given an order ideal [Groebner escalier of some ideal], there is a unique Bar Code associated to it.



$N = \{1, x_1, x_2, x_3\}$.
Viceversa $\rightarrow$ admissibility.

Bar codes and Pommaret basis

Given an order ideal, it is easy to recover the associated Pommaret basis from its Bar Code.



$N = \{1, x_1, x_2, x_3\}$; $\mathcal{F}(I) = \{x_1^2, x_1x_2, x_2^2, x_1x_3, x_2x_3, x_3^2\}$.

Remark:

A monomial ideal I is stable if and only if $\mathcal{F}(I) = G(I)$

(Strongly) stable ideals in two variables

The number of Bar Codes associated to (strongly) stable ideals in $k[x_1, x_2]$, whose bar list is (p, h), $p, h \in \mathbb{N}$, $p \geq h$ equals the number of **integer partitions** of p in h distinct parts, namely

$$p = \alpha_1 + ... + \alpha_h,\ \alpha_1 > ... > \alpha_h > 0.$$

$\rightarrow$ Simple formulas in literature.

Example

$p = 10 \rightarrow 10$ (strongly) stable ideals.

(Strongly) stable ideals in two variables

Denote by B a Bar Code associated to a (strongly) stable ideal $I \lhd k[x_1, x_2]$ with affine Hilbert polynomial $H_I(d) = p \in \mathbb{N}$ and by $\mathrm{L}_{\mathrm{B}} = (p, h)$ its bar list: then

$$1 \leq h \leq \lfloor \frac{-1 + \sqrt{1 + 8p}}{2} \rfloor$$

Example

For $p = 10$, $1 \leq h \leq 4$

Corollary

The number of Bar Codes associated to stable ideals in $k[x_1, ..., x_n]$, $n > 2$, whose bar list is $(p, h, \underbrace{1, ..., 1}_{3,...,n})$, $p, h \in \mathbb{N}$, $p \geq h$ equals the number of integer partitions of p in h distinct parts, namely

$$p = \alpha_1 + ... + \alpha_h,\ \alpha_1 > ... > \alpha_h > 0.$$

In three variables?

We need to consider the bar lists of the form (p, h, k), $k > 1$.

The *minimal sum* of a given list of positive integers $[\alpha_1, \dots, \alpha_g]$ is the integer $\mathbf{Sm}([\alpha_1, \dots, \alpha_g]) := \sum_{i=1}^{g} \frac{\alpha_i(\alpha_i+1)}{2}$

Example

$\mathrm{Sm}([1, 2, 3]) = 1 + (1 + 2) + (1 + 2 + 3) = 1 + 3 + 6 = 10$

Bounds

1. $k \in \{1, \dots, l\}$, where $l := \max_{i \in \mathbb{N}}\{i^3 + 3i^2 + 2i \leq 6p\}$;
2. $h \in \{\frac{k(k+1)}{2}, \dots, m\}$, where
   $m = \max_{r \geq \frac{k(k+1)}{2}} \{r \mid \exists \lambda \in I_{(r,k)}, \mathrm{Sm}(\lambda) \leq p\}$.

The number of stable ideals...

Theorem

There is a bijection between $P_{(p,h,k)}$ [plane partitions...] and the set $\mathsf{B}^{(S)}_{(p,h,k)}$ of Bar Codes B with $\mathsf{L}_{\mathsf{B}} = (p, h, k)$ and s.t. the corresponding order ideal is the Groebner escalier of a stable ideal.

$P_{(p,h,k)}$

$\overline{\beta} \in I_{(h,k)}$ [k rows, $\overline{\beta}$ 2-bars; norm p]:

$$\rho = (\rho_{i,j}) = \begin{pmatrix} \rho_{1,1} & \rho_{1,2} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \rho_{1,\overline{\beta_1}} \\ \rho_{2,1} & \cdots & \cdots & \cdots & \cdots & \cdots & \rho_{2,\overline{\beta_2}} & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \rho_{k,1} & \cdots & \cdots & \cdots & \cdots & \rho_{k,\overline{\beta_k}} & 0 & \cdots & \cdots \end{pmatrix}$$

s.t.

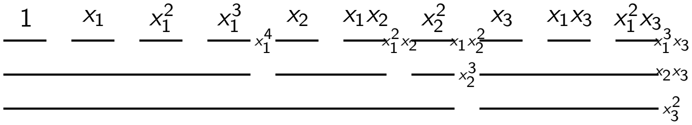
1. $\rho_{i,j} > 0$, $1 \leq i \leq k$, $1 \leq j \leq \overline{\beta_i}$;
2. $\rho_{i,j} > \rho_{i,j+1}$, $1 \leq i \leq k$, $1 \leq j \leq \overline{\beta_i} - 1$;
3. $\rho_{i,j} > \rho_{i+1,j}$ $1 \leq i \leq k-1$, $1 \leq j \leq \overline{\beta_{i+1}}$;
4. $n(\rho) = \sum_{i=1}^{k} \sum_{j=1}^{\overline{\beta_i}} \rho_{i,j} = p$

Example

Let $p = 10$: 29 stable ideals. For example, the plane partition

$$\begin{pmatrix} 4 & 2 & 1 \\ 3 & 0 & 0 \end{pmatrix}$$

uniquely determines the Bar Code



which corresponds to the stable ideal
$I = (x_1^4, x_1^2x_2, x_1x_2^2, x_2^3, x_1^3x_3, x_2x_3, x_3^2)$

Determinantal formulas

Theorem (Krattenthaler)

Let c, d be arbitrary integers, $\lambda, \mu \in D_r$ and let a, b be r-tuples of integers s.t. $a_i - c(\mu_i - \mu_{i+1}) + (1-d) \geq a_{i+1}$
$b_i + c(\lambda_i - \lambda_{i+1}) + (1-d) \geq b_{i+1}$ *for* $i = 1, 2, ..., r-1$.
$N_1(s,t) = b_s(\lambda_s - s - \mu_t + t) + (1-c-d)\left[\binom{\mu_t + s - t}{2} - \binom{\mu_t}{2}\right] + c\binom{\lambda_s - s - \mu_t + t}{2}$,
the polynomial

$$det_{1 \leq s,t \leq r}\left(x^{N_1(s,t)}\begin{bmatrix}(1-c)(\lambda_s - \mu_t) - d(s-t) + a_t - b_s + c \\ \lambda_s - s - \mu_t + t\end{bmatrix}\right),$$

is the norm generating function for (c,d)-plane partitions of shape λ/μ in which the first part in row i is at most a_i and the last part in row i is at least b_i.

Strongly stable ideals in three variables

Theorem

There is a bijection between $S_{(p,h,k)}$ *[shifted plane partitions...] and the set* $\mathrm{B}_{(p,h,k)}$ *of Bar Codes* B *with* $\mathrm{L_B} = (p, h, k)$ *and s.t. the corresponding order ideal is the Groebner escalier of a strongly stable ideal.*

$S_{(p,h,k)}$

$\overline{\alpha} \in I_{(h,k)}$,

$$\pi = (\pi_{i,j}) = \begin{pmatrix} \pi_{1,1} & \pi_{1,2} & \dots & \dots & \dots & \dots & \dots & \dots & \pi_{1,\overline{\alpha_1}} \\ 0\dots & \pi_{2,2} & \dots & \dots & \dots & \dots & \dots & \pi_{2,2+\overline{\alpha_2}-1} & 0\dots \\ 0\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0\dots & \dots & \dots & \pi_{k,k} & \dots & \dots & \pi_{k,k+\overline{\alpha_k}-1} & 0\dots & \dots \end{pmatrix}$$

s.t.

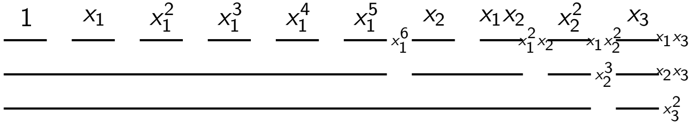
1. $\pi_{i,j} > 0,\ 1 \leq i \leq k,\ i \leq j \leq i + \overline{\alpha_i} - 1;$
2. $\pi_{i,j} > \pi_{i,j+1},\ 1 \leq i \leq k,\ i \leq j < i + \overline{\alpha_i} - 1;$
3. $\pi_{i,j} \geq \pi_{i+1,j}\ 1 \leq i \leq k-1,\ i+1 \leq j \leq i + \overline{\alpha_{i+1}} - 1;$
4. $n(\pi) = \sum_{i=1}^{k} \sum_{j=i}^{i+\overline{\alpha_i}-1} \pi_{i,j} = p.$

Example

Let $p = 10$: 24 strongly stable ideals. For example, the plane partition

$$\begin{pmatrix} 6 & 2 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

uniquely determines the Bar Code



which corresponds to the stable ideal
$I = (x_1^6, x_1^2x_2, x_1x_2^2, x_2^3, x_1x_3, x_2x_3, x_3^2)$

Determinantal formulas (II)

Let c, d be arbitrary integers, λ a partition with $\lambda_r \geq r$ and let a, b be r-tuples of integers satisfying

$$a_i - c - d \geq a_{i+1}$$

$$b_i + c(\lambda_i - \lambda_{i+1}) + (1 - d) \geq b_{i+1}$$

for $i = 1, 2, ..., r - 1$. $N_1 = \sum_{i=1}^{r}(b_i(\lambda_i - i) + a_i + c\binom{\lambda_i - i}{2})$, the polynomial

$$x^{N_1} det_{1 \leq s,t \leq r} \left(\begin{bmatrix} (\lambda_s - s)(1-c) + (1-c-d)(s-t) + a_t - b_s \\ \lambda_s - s \end{bmatrix} \right),$$

is the norm generating function for shifted (c, d)-plane partitions of shape λ in which the first part in row i is equal to a_i and the last part in row i is at least b_i.

$n \geq 4$

Let $(i_1, ..., i_n), (j_1, ..., j_n) \in \mathbb{N}^n$; we say that $(i_1, ..., i_n) < (j_1, ..., j_n)$ if $i_1 \leq j_1, ..., i_n \leq j_n$ but $(i_1, ..., i_n) \neq (j_1, ..., j_n)$.

Let $\rho = (\rho_{i,j})_{i \in \{1,...,r\}, j \in \{1,...,\beta_i\}}$ be a $(1,1)$-plane partition of shape $\beta = (\beta_1, ..., \beta_r)$, $\beta_1 > ... > \beta_r$. A *strict solid partition* (or *strict 3-partition*) of shape ρ is a 3-dimensional array $\gamma = (\gamma_{i_1,i_2,i_3})$, $1 \leq i_1 \leq \beta_{i_3}$, $1 \leq i_2 \leq \rho_{i_3,i_1}$, $1 \leq i_3 \leq r$, s.t.

- for each $1 \leq l \leq r$, the 2-dimensional array $\gamma_l := (\gamma_{i_1,i_2,l})$ is a $(1,1)$-plane partition of shape $\rho_l = (\rho_{l,1}, ..., \rho_{l,\beta_l})$.
- $\gamma_{i_1,i_2,i_3} > \gamma_{j_1,j_2,j_3}$, for $(i_1, i_2, i_3) < (j_1, j_2, j_3)$.

$n \geq 4$?

Let us consider the (1, 1)-plane partition of shape $\beta = (3, 2, 1)$

$$\rho = \begin{pmatrix} 4 & 2 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

An example of strict solid partition of shape ρ is is the following γ, formed by three (1, 1)-plane partitions $\gamma_1, \gamma_2, \gamma_3$:

$$\gamma_1 = \begin{pmatrix} \gamma_{\mathbf{1},\mathbf{1},\mathbf{1}} & \gamma_{\mathbf{1},\mathbf{2},\mathbf{1}} & \gamma_{1,3,1} & \gamma_{1,4,1} \\ \gamma_{\mathbf{2},\mathbf{1},\mathbf{1}} & \gamma_{2,2,1} & 0 & 0 \\ \gamma_{3,1,1} & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \mathbf{4} & \mathbf{3} & 2 & 1 \\ \mathbf{3} & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$\gamma_2 = \begin{pmatrix} \gamma_{\mathbf{1},\mathbf{1},\mathbf{2}} & \gamma_{1,2,2} & 0 \\ \gamma_{2,1,2} & 0 & 0 \end{pmatrix} = \begin{pmatrix} \mathbf{2} & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\gamma_3 = \begin{pmatrix} \gamma_{1,1,3} & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}$$

where we mark in bold the elements of γ_i over which those of γ_{i+1} are posed, for $i = 1, 2$.

$n \geq 4$?

For $n \geq 4$, consider a strict $(n-1)$-partition $\rho = (\rho_{\bar{i}_1,\ldots,\bar{i}_{n-1}})$ with $1 \leq \bar{i}_{n-1} \leq h$, for some $h > 0$.
A *strict n-partition* of shape ρ is a n-dimensional array $\gamma = (\gamma_{i_1,\ldots,i_n})$ s.t.

- for each $1 \leq l \leq h$, $\gamma_l := (\gamma_{i_1,\ldots,i_{n-1},l})$ is a strict $(n-1)$-partition of shape $\rho_l = (\rho_{\bar{i}_1,\ldots,\bar{i}_{n-2},l})$
- $\gamma_{i_1,\ldots,i_n} > \gamma_{j_1,\ldots,j_n}$, for $(i_1,\ldots,i_n) < (j_1,\ldots,j_n)$.

$n \geq 4$?

Let us consider the following very simple strict solid partition ρ:

$$\rho_1 = \begin{pmatrix} \mathbf{2} & 1 \\ 1 & 0 \end{pmatrix} \quad \rho_2 = \begin{pmatrix} 1 & 0 \end{pmatrix}$$

An example of strict 4-partition of shape ρ is

$$\gamma_1 = \begin{pmatrix} \gamma_{\mathbf{1,1,1,1}} & \gamma_{1,2,1,1} \\ \gamma_{2,1,1,1} & 0 \end{pmatrix} \quad \begin{pmatrix} \gamma_{1,1,2,1} & 0 \end{pmatrix} = \begin{pmatrix} \mathbf{4} & 2 \\ 2 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 \end{pmatrix}$$

$$\gamma_2 = \begin{pmatrix} \gamma_{1,1,1,2} & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \end{pmatrix}$$

Conjecture 1

There is a bijection between the set $P_\rho(p_1, ..., p_n)$ [strict n-partitions] and the set

$$B_{(p_1,...,p_n)} := \{\mathsf{B} \in A_n \text{ s.t. } \mathsf{L_B} = (p_1, ..., p_n),\ \eta(\mathsf{B}) = \mathsf{N}(J),\ J \text{ stable}\}.$$

And with shifts

Let $\pi = (\pi_{i,j})_{i \in \{1,\dots,r\}, j \in \{1,\dots,\alpha_i\}}$ be a shifted $(1,0)$-plane partition of shape $\alpha = (\alpha_1, \dots, \alpha_r)$, $\alpha_1 \geq \dots \geq \alpha_r \geq r$. A *shifted solid partition* (or *shifted* 3-*partition*) of shape π is a 3-dimensional array $\gamma = (\gamma_{i_1,i_2,i_3})$, $i_3 \leq i_1 \leq \alpha_{i_3}$, $i_1 \leq i_2 \leq \pi_{i_3,i_1} + i_1 - 1$, $1 \leq i_3 \leq r$, s.t.

- for each $1 \leq l \leq r$, the 2-dimensional array $\gamma_l := (\gamma_{i_1,i_2,l})$ is a shifted $(1,0)$-plane partition of shape $\widetilde{\pi}_l = (\pi_{l,l} + l - 1, \pi_{l,l+1} + l, \dots, \pi_{l,\alpha_l} + \alpha_l - 1)$.
- $\gamma_{i_1,i_2,i_3} \geq \gamma_{i_1,i_2,i_3+1}$.

Example

Let us consider the shifted $(1,0)$-plane partition

$$\pi = \begin{pmatrix} 4 & 2 & 1 \\ 0 & 2 & 0 \end{pmatrix}$$

of shape $\alpha = (3,2)$.
An example of strict solid partition of shape π is is the following γ, formed by two shifted $(1,0)$-plane partitions γ_1, γ_2:

$$\gamma_1 = \begin{pmatrix} \gamma_{1,1,1} & \gamma_{1,2,1} & \gamma_{1,3,1} \\ 0 & \gamma_{\mathbf{2},\mathbf{2},1} & \gamma_{\mathbf{2},\mathbf{3},1} \\ 0 & 0 & \gamma_{3,3,1} \end{pmatrix} = \begin{pmatrix} 3 & 2 & 1 \\ 0 & \mathbf{2} & \mathbf{1} \\ 0 & 0 & 1 \end{pmatrix}$$

$$\gamma_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \gamma_{2,2,2} & \gamma_{2,3,2} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 1 \end{pmatrix}$$

where we mark in bold the elements of γ_1 over which those of γ_2 are posed.

n-TH SHIFT

For $n \geq 4$, consider a shifted $(n-1)$-partition $\pi = (\pi_{\bar{i}_1,\ldots,\bar{i}_{n-1}})$ with $1 \leq \bar{i}_{n-1} \leq h$, for some $h > 0$.
A *shifted n-partition* of shape π is a n-dimensional array $\gamma = (\gamma_{i_1,\ldots,i_n})$ s.t.

- for each $1 \leq l \leq h$, $\gamma_l := (\gamma_{i_1,\ldots,i_{n-1},l})$ is a shifted $(n-1)$-partition with shape given by the $(n-2)$-partition $\widetilde{\pi}_l = (\pi_{\bar{i}_1,\ldots,\bar{i}_{n-2},l} + i_m - 1)$, where m is the maximal index s.t. $i_m > 1$ and such that, w.r.t. the ordering defined above, the minimal $(i_1, \ldots, i_{n-1}, i_n)$ for which $\gamma_{i_1,\ldots,i_{n-1},i_n} \neq 0$ s.t. $i_n = l$ is $(l, l, \ldots, l)$;
- $\gamma_{i_1,\ldots,i_n} \geq \gamma_{i,\ldots,i_n+1}$.

Example

Let us consider the following very simple shifted solid partition π:

$$\pi_1 = \begin{pmatrix} 2 & 1 \\ 0 & \mathbf{1} \end{pmatrix} \quad \pi_2 = \begin{pmatrix} 0 & 1 \end{pmatrix}$$

An example of strict 4-partition of shape π is

$$\gamma_1 = \begin{pmatrix} \gamma_{1,1,1,1} & \gamma_{1,2,1,1} \\ 0 & \gamma_{\mathbf{2},\mathbf{2},\mathbf{1},\mathbf{1}} \end{pmatrix} \quad \begin{pmatrix} 0 & 0 \\ 0 & \gamma_{\mathbf{2},\mathbf{2},\mathbf{2},1} \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 0 & \mathbf{2} \end{pmatrix} \quad \begin{pmatrix} 0 & 0 \\ 0 & \mathbf{1} \end{pmatrix}$$

$$\gamma_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 \\ 0 & \gamma_{2,2,2,2} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

There is a bijection between the set $S_\pi(p_1, ..., p_n)$ [shifted n-partitions] and the set

$$B_{(p_1,...,p_n)} := \{\mathsf{B} \in A_n \text{ s.t. } \mathsf{L_B} = (p_1, ..., p_n),\ \eta(\mathsf{B}) = \mathsf{N}(J),\ J \text{ strongly stable}\}.$$

Thank you for your attention!