On the existence of birational surjective parametrizations of affine surfaces

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A first interesting question is "how is the part of S that f does not cover?"
The answer: No more than a finite amount of rational curves and an isolated point.
There are also algorithms to find an Atlas of birational maps for the whole surface.

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Main idea: Take a smooth cubic surface S whose intersection with the infinity hyperplane S_∞ is smooth (so elliptic). Let $f : \mathbb{P}^2 \dashrightarrow S$ a birational parametrization.

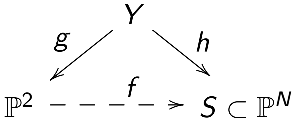
The image of the infinity line cannot be S_∞, so it seems that such image should not be reached by $\mathbb{A}^2$ (by Zariski's Main Theorem, since S is smooth).

We could end the talk here, but let us give more details...

For the general set up, recall that

- any map defined in an open subset of $\mathbb{P}^2$ can be extended to the complement of a finite subset;
- then, indeterminacies can be solved by blowing up such points as many times as we need

So we have the commutative diagram, where g is just a sequence of blowups and h is a morphism:



Let L_∞ be $\mathbb{P}^2 - \mathbb{A}^2$, $S_\infty = S - \mathbb{A}^N$ and $F(f)$ be the fundamental locus of f.

Now suppose that $f(\mathbb{A}^2)$ contains all $S \cap \mathbb{A}^N$...

$$\begin{array}{ccc} & Y & \\ {\scriptstyle g}\swarrow & & \searrow{\scriptstyle h} \\ \mathbb{P}^2 & \overset{f}{\dashrightarrow} & S \subset \mathbb{P}^N \end{array}$$

If $f(L_\infty)$ is a curve, it is rational.
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Theorem (Zariski's Main Theorem)

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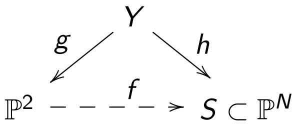

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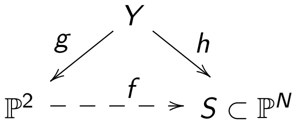
Then, if $f(L_\infty)$ is a curve and S is normal, it is a rational component of S_∞.

$S \cap \mathbb{A}^N \subset f(\mathbb{A}^2)$, f contracts L_∞



Theorem (Castelnuovo's criterion of contractibility)

If L_∞ is contracted to a smooth point, the proper transform of L_∞ by g is a component of the exceptional divisor for h, and it is then a curve with negative selfintersection.

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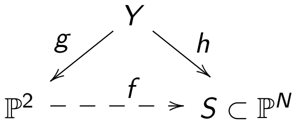

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This is only possible if g blows at least one point in L_∞ up.

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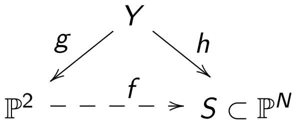

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Theorem (Zariski's Main Theorem)

$h(g^{-1}(P))$ consists of a finite amount of rational curves.

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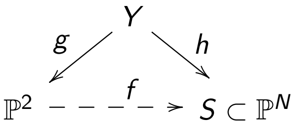

For the general $Q \in h(g^{-1}(P))$, $h^{-1}(Q)$ is a zero-dimensional set.

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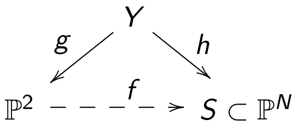

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If Q is normal, since h is birational, $h^{-1}(Q)$ is a point.

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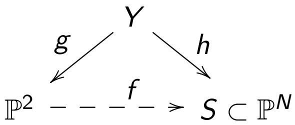

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If Q is normal, since h is birational, $h^{-1}(Q)$ is a point.

Then $Q \notin f(\mathbb{A}) \Rightarrow Q \in S_\infty$.

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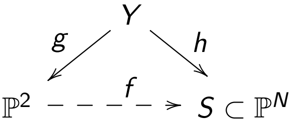

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Theorem (Zariski's Main Theorem)

If Q is normal, since h is birational, $h^{-1}(Q)$ is a point.

Then $Q \notin f(\mathbb{A}) \Rightarrow Q \in S_\infty$.
Therefore: $h(g^{-1}(F(f))) \subset S_\infty$ when S is normal.

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Summary:

- If L_∞ is not contracted and S is normal, then $f(L_\infty)$ is a rational component of S_∞

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- If L_∞ is not contracted and S is normal, then $f(L_\infty)$ is a rational component of S_∞
- If L_∞ is contracted and S is smooth, then at least one point of L_∞ is fundamental for f and it provides at least one rational component for S_∞.

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Summary:

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So:

Theorem

Let $f : \mathbb{C}^2 \dashrightarrow \mathbb{C}^N$ be a rational map. Let S be the Zariski closure of $f(\mathbb{C}^2)$ in $\mathbb{C}^N$, and suppose that f is birational and surjective onto S. Let $\overline{S}$ be the Zariski closure of S in $\mathbb{P}^N$ and $S_\infty = \overline{S} - S$ the infinite hyperplane section. If $\overline{S}$ is smooth, then S_∞ has at least one rational component.

Easy consequence

If S is smooth and none of the components of S_∞ is rational, the affine subset $S \cap \mathbb{A}^N$ cannot be parametrized birationally and surjectively.

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This is the case of Fermat's cubic surface.

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This gives a reducible curve H that can be taken to the infinity hyperplane by composing a Veronese map with a linear transformation on $\mathbb{P}^N$.

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- Since $h(g^{-1}(F(f))) \subset S_\infty$, Zariski's Main Theorem says that then f is a morphism.
- But that is impossible (Bezout's thorem, compose with projection from S to $\mathbb{P}^1$...)

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- Higher dimension (Glups!).
- Fields that are not algebraically closed (especially $\mathbb{R}$) (More glupses)

Thanks for your attention!