Fundamental Operations on Rank Metric Codes

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MEGA
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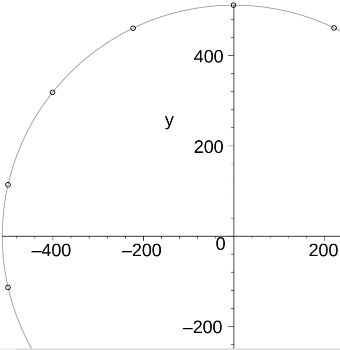
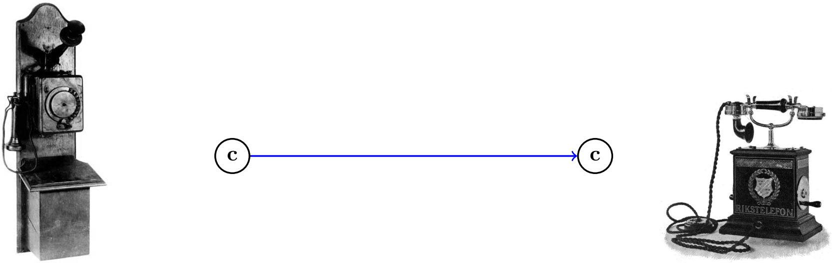


What is Coding Theory About?

Coding Theory was introduced after Shannon's noisy channel theorem (1948) for efficient communication across noisy channels.



sender transmits **c**, receiver gets **c**

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sender transmits $\mathbf{c}$, receiver gets $\mathbf{c}+\mathbf{e}=\mathbf{v}$

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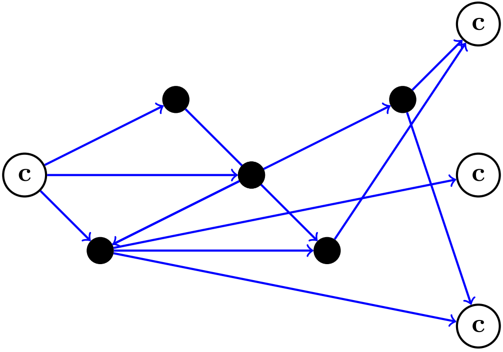


sender transmits **c**, receivers want **c**

Encoding

- $m \in \mathbb{F}_q^k$ is a message
- encode m by multiplication with a full-rank $k \times n$ matrix

$$G : \mathbb{F}_q^k \longrightarrow \mathbb{F}_q^n : m \mapsto c = mG$$

$$C = \{mG : m \in \mathbb{F}_q^k\}$$

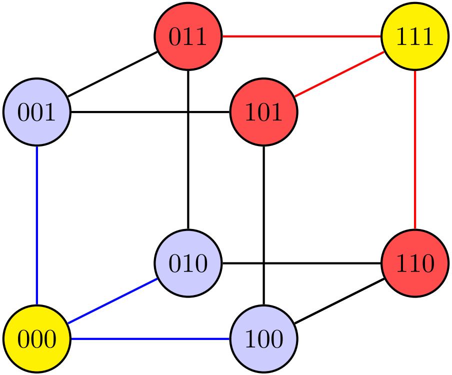
is an $\mathbb{F}_q$-$[n, k, d]$ code.

What is d?

d is the minimum distance between a pair of distinct codewords.

Want low n, high k, high d.
The higher d is, the more robust the code is to noise (packing problem).

Sphere-Packing



Fundamentals in Coding Theory

- Operations on codes - making new codes from old:
  - puncturing,
  - shortening,
  - extending,
  - concatenating,
  - products,
- Parameters of codes
  - length,
  - dimension,
  - minimum distance,
  - packing radius,
  - covering radius,
  - weight enumerators.
- Weight enumerators
  - MacWilliams duality theorem,
  - the zeta function.

q-Analogues

subsets $\{s_1, \ldots, s_k\}$ of $[n]$	subspaces $\langle s_1, \ldots, s_k \rangle$ of $\mathbb{F}_q^n$
set cardinality	vector space dimension
binomial coefficients $\binom{n}{k}$	Gaussian coefficients $\begin{bmatrix} n \\ k \end{bmatrix}_q$
Hamming weight of $(v_1, \ldots, v_n) \in \mathbb{F}_{q^m}^n$	$\mathbb{F}_q$-dimension of $\langle v_1, \ldots, v_n \rangle \subset \mathbb{F}_{q^m}^n$
Hamming weight of $(v_1, \ldots, v_n) \in \mathbb{F}_{q^m}^n$	$\mathbb{F}_q$-rank of $\begin{pmatrix} v_{11} & v_{12} & \cdots & v_{1n} \\ v_{21} & v_{22} & \cdots & v_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ v_{m1} & v_{m2} & \cdots & v_{mn} \end{pmatrix} \in \mathbb{F}_q^{m \times n}$

Rank Metric Codes

- Introduced by Delsarte (1978) as a q-analogue of coding theory.
- Independently introduced by Gabidulin (1986) and Roth (1991) for array error correction.
- Studied more after 2000 in the context of code-based-cryptosystems.
- Since 2008, generated interest among algebraic coding theorists due to their applicability in network error correction.
- Many open problems in coding theory: only since 2015 have we seen new optimal families of rank metric codes.

Hamming Metric Codes

Definition 1

A linear $\mathbb{F}_q$-$[n,k,d]$ code C is a k-dimensional subspace of $\mathbb{F}_q^n$ of minimum Hamming distance

$$d = \min\{d_H(c,c') : c,c' \in \mathbb{C}\}.$$

- $d_H((u_1,...,u_n),(v_1,...,v_n)) := |\{i \in [n] : u_i \neq v_i\}|$.
- C is optimal if k attains the max. possible dimension for fixed n,d.

Theorem 2 (Singleton Bound, 1964)

If C is an $[n,k,d]$ code then $k \leq n-d+1$.

- Codes that meet the Singleton bound are called **maximum distance separable** (MDS).
- MDS code exist for $n \leq q+1$ (via Reed-Solomon codes).
- Segre (1955) conjectured that if $k \leq q$, q odd then $n \leq q+1$.

Rank-Metric Codes

Definition 3

A linear $\mathbb{F}_q$-$[m \times n, k, d]$ rank-metric code C is a k-dimensional subspace of $\mathbb{F}_q^{m \times n}$ of minimum rank distance

$$d = \min\{\mathrm{rk}(A - B) : A, B \in C\}.$$

- rk is a distance function on $\mathbb{F}_q$-$[m \times n, k, d]$.
- C is optimal if k attains the max. possible dimension for fixed m, n, d.

Theorem 4 (Rank Singleton Bound, Delsarte 1978)

If C is an $[m \times n, k, d]$ code with $n \leq m$ then $k \leq m(n - d + 1)$.

- Codes that meet the rank Singleton bound are called **maximum rank distance** codes (MRD).
- MRD codes exist for all m, n, d.

A Construction of MDS Codes

The Reed-Solomon Codes form a class of MDS codes.
Choose $\alpha_1, ..., \alpha_n$ distinct in $\mathbb{F}_q^\times$.

$$\mathrm{RS}(n,k) := \{c_f = (f(\alpha_1), ..., f(\alpha_n)) : f \in \mathbb{F}_q[x], \deg(f) \leq k-1\}$$

Any pair of distinct polynomials $f, g \in \mathbb{F}_q[x]$ of degree $\leq k-1$ have at most $k-2$ common roots so

$$d_H(c_f, c_g) \geq n-k+1.$$

From the Singleton bound its minimum distance is $\leq n-k+1$, so $\mathrm{RS}(n,k)$ is MDS.
Remark: For a basis-free approach, identify $\mathrm{RS}(n,k)$ with

$$\{f \in \mathbb{F}_q[x], \deg(f) \leq k-1\}.$$

A Construction of MRD Codes

The Delsarte-Gabidulin Codes form a class of MRD codes (1978, 1984).

- $L_m := \{f_0x + f_1x^q + \cdots + f_kx^{q^{m-1}} : f_i \in \mathbb{F}_{q^m}\}$ (linearized polynomials in $\mathbb{F}_{q^m}[x]$)
- Choose $\alpha_1, ..., \alpha_n \subset \mathbb{F}_{q^m}$, linearly independent over $\mathbb{F}_q$

$$\mathscr{G}(m,n,k) := \{c_f = (f(\alpha_1), ..., f(\alpha_n)) : f \in L_m, \deg(f) \leq q^{k-1}\}$$

$$f, g \in L_m, \deg f, \deg g \leq q^{k-1} \implies \dim(\ker f \cap \ker g) \leq k-1 \implies$$
$$d_{\mathrm{rk}}(c_f, c_g) \geq n-k+1.$$

- $\mathscr{G}(n,k)$ is MRD by rank Singleton bound.
- $M_{n\times n}(\mathbb{F}_q) \cong L_n$ as rings (multiplication modulo $x^{q^n} - 1$).

For a basis-free approach, define

$$\mathscr{G}(m,k) := \{f_0x + f_1x^q + \cdots + f_kx^{q^{k-1}} : f_i \in \mathbb{F}_{q^m}\}.$$

MRD Codes

- Delsarte-Gabidulin codes admit fast decoding.
- Until 2015 they were the only known family of MRD codes.

The twisted Delsarte-Gabidulin codes were discovered by Sheekey in 2015.

$$\mathscr{H}(n,k) := \{f_0 x + f_1 x^q + \cdots + f_k x^{q^{k-1}} + f_0 \theta^{q^h} x^{q^k} : f_i \in \mathbb{F}_{q^n}\}.$$

Theorem 5

If $\theta^{\frac{q^n-1}{q-1}} \neq (-1)^{nk}$ then $\mathscr{H}(n,k)$ is MRD with parameters $[n \times n, kn, n-k+1]$

- The converse is false.
- The Delsarte-Gabidulin codes are $\mathbb{F}_{q^n}$-linear.
- The twisted Delsarte-Gabidulin codes are not always $\mathbb{F}_{q^n}$-linear.
- Few other families of rank-metric codes are known.
- Most MRD codes are not twisted Delsarte-Gabidulin codes.

The Hamming Weight Enumerator

The weight of a codeword is its distance to zero, wrt a given distance function. Given a linear code $C \subset \mathbb{F}_q^n$, its Hamming weight enumerator is

$$W(x,y) = \sum_{i=0}^{n} W_i x^{n-i} y^i,$$

where $W_t := |\{c \in C : d_H(c,0) = t\}|$ for $0 \leq t \leq n$.

The dual $C^\perp := \{v \in \mathbb{F}_q^n : c \cdot v = 0 \, \forall c \in C\}$ has weight enumerator $W^\perp(x,y)$ st:

Theorem 6 (MacWilliams Duality Theorem)

$$W^\perp(x,y) = \frac{1}{|C|} W(x+(q-1)y, x-y)$$

The Rank Weight Enumerator

Given a linear code $C \subset \mathbb{F}_q^{m \times n}$, its rank weight enumerator is

$$W(x,y) = \sum_{i=0}^{n} W_i x^{n-i} y^i,$$

where $W_t := |\{X \in C : \operatorname{rk} X = t\}|$ for $0 \leq t \leq n$.

In 2008 Gadouleau and Yan derived the q-analogue of the MacWilliams duality theorem. (Also Delsarte 1970s via association schemes)

$C^\perp := \{Y \in \mathbb{F}_q^{m \times n} : \operatorname{tr}(XY^T) = 0 \; \forall X \in C\}$ has weight enumerator $W^\perp(x,y)$ st:

Theorem 7 (Rank metric duality theorem)

$$W^\perp(x,y) = \frac{1}{|C|} \tilde{W}(x + (q^m - 1)y, x - y)$$

where $\tilde{W}(x,y)$ is a q-transform of $W(x,y)$.

Weight Enumerators

The weight enumerator is an important invariant of a code.

For example, weight enumerators relate codes to designs, strongly regular graphs and association schemes.

It also tells us precisely how effective the code is for transmitting information.

- For some extremal codes, the weight enumerator is determined.
- In the Hamming metric, this occurs for MDS codes.
- In the rank metric, this occurs for MRD codes.
- The MDS/MRD property of a weight enumerator is invariant under puncturing and shortening.
- The MDS/MRD weight enumerators are $\mathbb{Q}$-bases for the spaces of Hamming/rank metric weight enumerators.

Puncturing Hamming Metric Codes

Puncture an $[n,k,d]$ code in $\mathbb{F}_q^n$ by deleting the same coord. from each codeword. If $d>1$ this results in an $[n-1,k,\geq d-1]$ code.

Example 8

Puncture an $\mathbb{F}_2$-$[8,4,4]$ code on the last coordinate to get an $\mathbb{F}_2$-$[7,4,3]$ code.

$$\begin{array}{cc} 00000000 & 11111111 \\ 11100001 & 00011110 \\ 10011001 & 01100110 \\ 10000111 & 01111000 \\ 01010101 & 10101010 \\ 01001011 & 10110100 \\ 00110011 & 11001100 \\ 00101101 & 11010010 \end{array} \rightarrow \begin{array}{cc} 0000000 & 1111111 \\ 1110000 & 0001111 \\ 1001100 & 0110011 \\ 1000011 & 0111100 \\ 0101010 & 1010101 \\ 0100101 & 1011010 \\ 0011001 & 1100110 \\ 0010110 & 1101001 \end{array}$$

The punctured code has a better rate, but worse minimum distance.

Shortening Hamming Metric Codes

Shorten an $[n,k,d]$ code by choosing the subcode with zero entries in a given coordinate and then deleting that same coordinate from each selected codeword. If $d>1$ this results in an $[n-1,k-1,\geq d]$ code.

Example 9

Shorten an $\mathbb{F}_2$-$[8,4,4]$ code on the last coordinate to get an $\mathbb{F}_2$-$[7,3,4]$ code.

$$\begin{array}{cc} 00000000 & 11111111 \\ 11100001 & 00011110 \\ 10011001 & 01100110 \\ 10000111 & 01111000 \\ 01010101 & 10101010 \\ 01001011 & 10110100 \\ 00110011 & 11001100 \\ 00101101 & 11010010 \end{array} \rightarrow \begin{array}{cc} 0000000 & \\ & 0001111 \\ & 0110011 \\ & 0111100 \\ & 1010101 \\ & 1011010 \\ & 1100110 \\ & 1101001 \end{array}$$

The shortened code has a worse rate, but may have a higher minimum distance.

Shortening and Puncturing Rank-Metric Codes

We define shortening/puncturing as projections to $\mathbb{F}_q^{m\times(n-1)}$.

Definition 10

Let $H \in \mathbb{F}_q^{n\times(n-1)}$ have rank $n-1$. Let $h \in \mathbb{F}_q^n \backslash \mathrm{col}(H)$.
The punctured and shortened codes of C wrt H are:

$$\begin{aligned} \Pi_H(C) &:= \{XH : X \in C\} \subset \mathbb{F}_q^{m\times(n-1)} \text{ (punctured code)}, \\ \Sigma_{h,H}(C) &:= \{XH : X \in C, Xh^T = 0\} \subset \mathbb{F}_q^{m\times(n-1)} \text{ (shortened code)}. \end{aligned}$$

Example 11

Let $E_i = [e_j^T : j \neq i]$, $e_j = [0,\dots,1,\dots,0]$.

- $\Pi_{E_i}(C)$: delete the ith col of each elt of C.
- $\Sigma_{e_i,E_i}(C)$: delete the ith col of each elt of C whose ith col is zero.

Shortening and Puncturing Rank-Metric Codes

Example 12

Here's an $\mathbb{F}_2$-$[4 \times 4, 3, 4]$ code, C.

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

Then $\Sigma_{e_1,E_1}(C) = \{0\}$ as C has weight enumerator $x^4 + 7xy^3$.

$Xh^T \neq 0$ for any $h \neq 0$, so all shortened codes of C are trivial.

Shortening and Puncturing Rank-Metric Codes

Example 13

Here's an $\mathbb{F}_2$-$[3 \times 3, 4, 2]$ code with $W(x,y) = x^3 + 13xy^2 + 2y^3$.

$$C = \left\langle \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \right\rangle.$$

H	h	$W_{\Sigma_{h,H}}(x,y)$	H	h	$W_{\Sigma_{h,H}}(x,y)$
E_2	110	$x^2 + y^2$	E_3	001	$x^2 + 3y^2$
	010	$x^2 + y^2$		111	$x^2 + y^2$
	111	$x^2 + y^2$		011	$x^2 + y^2$
	011	$x^2 + y^2$		101	$x^2 + 3y^2$

Duality of Puncturing and Shortening in the Hamming Metric

- The MDS property $(k = n - d + 1)$ is invariant under shortening and puncturing.
- Puncturing is really a projection to $\mathbb{F}_q^{n-1}$.
- Shortening is projection of a subcode to $\mathbb{F}_q^{n-1}$.
- We can puncture/shorten on several coords.

Recall $C^\perp := \{x \in \mathbb{F}_q^n : c \cdot x = 0 \, \forall c \in C\}$.

Theorem 14 (Duality of Puncturing and Shortening)

Let $\mathbf{P}_i(C)$ and $\mathbf{S}_i(C)$ be the punctured and shortened codes of C at the ith coordinate, respectively. Then

$$\mathbf{P}_i(C)^\perp = \mathbf{S}_i(C^\perp).$$

Pf: (Easy) $\mathbf{S}_i(C^\perp) \subset \mathbf{P}_i(C)^\perp$. Show equality by comparing dimensions.

Duality of Puncturing and Shortening in the Rank Metric

Recall for an $\mathbb{F}_q$-$[m \times n, k, d]$ code C,

$$C^\perp := \{N \in \mathbb{F}_q^{m\times n} : \mathrm{Tr}(MN^t) = 0 \text{ for all } M \in C\} \subseteq \mathbb{F}_q^{m\times n}.$$

Theorem 15 (B., Ravagnani 2016)

Duality of puncturing and shortening also holds for rank metric codes. In particular,

$$\Pi_{E_i}(C)^\perp = \Sigma_{e_i,E_i}(C^\perp).$$

- $k^\perp := \dim(C^\perp) = mn - k$
- $C^{\perp\perp} = C$
- If C is not $\mathbb{F}_{q^m}$-linear, its duals under the trace inner product and the scalar inner product are different.

Parameters of Punctured and Shortened Codes

Lemma 16 (Hamming Metric)

Let C be an $\mathbb{F}_q$-$[n,k,d]$ code.

1. $\mathbf{P}_i(C)$ *is* $[n-1, k, \geq d-1]$
2. $\mathbf{S}_i(C)$ *is* $[n-1, k-1, \geq d]$.
3. *If C is MDS then so is* $\mathbf{P}_i(C)$.
4. *If C is MDS then so is* $\mathbf{S}_i(C)$.

Lemma 17 (Rank Metric)

Let C be an $\mathbb{F}_q$-$[m \times n, k, d]$ code. Let $H \in \mathbb{F}_q^{n\times(n-1)}$ have rank $n-1$ with $h \notin \mathrm{col}(H)$.

1. $\Pi_H(C)$ *is* $[m \times (n-1), k, \geq d-1]$
2. $\Sigma_{h,H}(C)$ *is* $[m \times (n-1), \geq k-m, \geq d]$.
3. *If C is MRD then so is* $\Pi_H(C)$.
4. *If C is MRD then so is* $\Sigma_{h,H}(C)$.

The Zeta Function of a Curve

- $\mathscr{C}$ non-singular projective curve over $\mathbb{F}_q$,
- N_k the number of $\mathbb{F}_{q^k}$-rational points of $\mathscr{C}$,

The zeta-function of $\mathscr{C}$ is

$$Z(\mathscr{C}, T) = exp\left(\sum_{k\geq 1} \frac{N_k}{k} T^k\right).$$

Theorem 18 (Weil, Dwork)

The zeta function of any non-singular projective curve of genus g can be expressed as

$$Z(\mathscr{C}, T) = \frac{P(T)}{(1-T)(1-qT)},$$

some $P(T) \in \mathbb{Q}[T]$ $\deg P(T) \leq 2g$. $|\omega| = q^{-1/2}$ for each root ω of $P(T)$.

Zeta Functions for Hamming-Metric Codes

Definition 19 (Duursma 1999)

The **zeta polynomial** of a (Hamming metric) $\mathbb{F}_q$-$[n,k,d]$ code C is the unique polynomial $P(T)$ of degree at most $n-d+1$ such that

$$\frac{P(T)}{(1-T)(1-qT)}(Tx+(1-T)y)^n = \cdots + \frac{W(x,y)-x^n}{q-1}T^{n-d} + \cdots .$$

The quotient

$$Z(T) := \frac{P(T)}{(1-T)(1-qT)}$$

is called the **zeta function** of C.

- The weight enumerator $M_{n,d}$ of an $\mathbb{F}_q$-$[n,d]$ MDS code is determined.
- $P(T) = \sum_{i=0}^{n-d+1} p_i T^i \implies W(x,y) = \sum_{i=0}^{n-d} p_i M_{n,d+i}(x,y) + p_{n-d+1}x^n$.
- If C is MDS then $P(T) = 1$.

MRD Weight Enumerators

- If C is MRD, then its weight enumerator is determined (Delsarte, 1978).

$$M_{m\times n,d}(x,y) \quad = \quad x^n + \sum_{i=d}^{n}(q^{m(i-d+1)}-1)\begin{bmatrix} n \\ i \end{bmatrix} y^i \prod_{t=0}^{n-i-1}(x-q^t y).$$

- The MRD weight enumerators

$$\left\{M_{m\times n,d}(x,y) : 0 \leq d \leq n\right\} \cup \{x^n\}$$

  are a $\mathbb{Q}$-**basis** for the space of all $m \times n$ 'weight enumerators' (homog. polys of degree n).
- Given any $[m\times n, k, d]$ code C, there exist unique coefficients $p_i \in \mathbb{Q}$ s.t. for some r,

$$W(x,y) = p_0 M_{m\times n,d}(x,y) + \cdots + p_r M_{m\times n,d+r}(x,y).$$

- The $p_0, \ldots, p_r$ turn out to coincide with the coefficients of the **zeta polynomial** of C.

Zeta Functions, Zeta Polynomials, Weight Enumerators

Theorem 20 (B., Blanco-Chacón, Duursma, Sheekey, 2017)

$$Z(T)\phi_n(T) = \frac{P(T)\phi_n(T)}{(1-T)(1-q^m T)} = \cdots + \frac{W(x,y)-x^n}{q^m-1} T^{n-d} + \cdots .$$

$P(T)$ is the unique polynomial of degree at most $n-d+1$ such that

$$W(x,y) = \sum_{i=0}^{n-d} p_i M_{m\times n, d+i}(x,y) + p_{n-d+1} x^n.$$

- $\phi_{n,r} = \begin{bmatrix} n \\ r \end{bmatrix} \prod_{j=0}^{r-1} (x - q^j y) y^{n-r}$,
- $\phi_n(T) := \sum_{r=0}^{n} \phi_{n,r}(x,y) T^r$,
- if C is MRD then $P(T) = 1$.

Zeta Functions

- $Z(\mathscr{C}, T)$ is the generating function for the number of points on a curve.
- $Z(T)$ is the generating function of **binomial moments** of a code.
- The binomial moments measure the average size of the **shortened subcodes**.
- The property $|\omega| = q^{-1/2}$ for every root ω of the zeta polynomial is called the Riemann hypothesis (RH).
- Many infinite families of codes with extremal Hamming weight enumerators sat. RH.
- It is conjectured that a sufficient condition for RH of a formally self-dual Hamming metric code is that it has weight distribution close to a random code.

Question 1

Which families of rank-metric codes satisfy the Riemann hypothesis?

$$(|\omega| = q^{-m/2}?)$$

The Riemann Hypothesis for Rank Metric Codes

Example 21

Any MRD code satisfies RH - it has $P(T) = 1$!

Example 22

- Take a (Hamming metric) extended binary QR code in $\mathbb{F}_2^{18}$.
- Puncture and shorten this code to get a code in $\mathbb{F}_2^{16}$.
- Express each resulting word in $\mathbb{F}_2^{16}$ as a 4×4 matrix.

The binomial moments are

$$b_0 = 0, b_1 = 0, b_2 = 3/5, b_3 = 15, b_4 = 255$$

$$P(T) = (1 + 8T + 16T^2)/25 = (1 + 4T)^2/25.$$

The zeroes $T = -1/4$ have absolute value $(2^4)^{-1/2} = 1/\sqrt{16}$ and so satisfy RH.

The zeta polynomial is that of a maximal elliptic curve over $\mathbb{F}_{16}$.

The Riemann Hypothesis for Rank Metric Codes

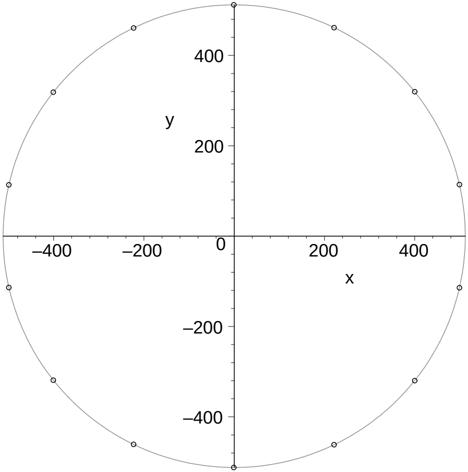


Figure: Complex zeroes for $P(T)$ of $\mathbb{F}_4^{9\times 9}$ in $\mathbb{F}_4^{18\times 18}$.

A Bound on the Minimum Distance

Theorem 23 (B., Blanco-Chacón, Duursma, Sheekey, 2017)

Let $P(T)$ be the zeta polynomial of an $\mathbb{F}_q$-$[m \times n, k, d]$ rank metric code and let θ be the negative of the sum of its reciprocal roots. Then

$$d \leq \log_q [(\theta + q^m + 1)(q-1) + 1] - 1.$$

- Follows due to MacWilliam's duality theorem for rank metric codes.

Shortened Subcodes, Binomial Moments and $W(x,y)$

Definition 24 (The shortened subcode of C)

We define the **shortened subcode** of C wrt $U \subseteq \mathbb{F}_q^n$ as:

$$C_U := \left\{ X \in C : Xu^T = 0 \ \ \forall u \in U \right\}.$$

Definition 25 (The Binomial Moments of C)

$$b_r = \begin{cases} \begin{bmatrix} n \\ \dim U \end{bmatrix}^{-1} \displaystyle\sum_{\dim U = n-d-r} (|C_U| - 1) & \text{if } 0 \leq r \leq n-d \\ 0 & \text{if } r < 0 \\ q^{k-mu} - 1, u = n-d-r & \text{if } r > n - d^{\perp} - d \end{cases}$$

Theorem 26 (B., Blanco-Chacón, Duursma, Sheekey, 2017)

$W(x,y)$ is completely determined by the b_i.

Shortened Subcodes, Binomial Moments and $W(x,y)$

Example 27

Here's an $\mathbb{F}_2$-$[3\times 3,4,2]$ code with $W(x,y)=x^3+13xy^2+2y^3$ and $d^\perp=1$.

$$C=\left\langle \begin{bmatrix}1&0&0\\0&0&0\\0&1&0\end{bmatrix},\begin{bmatrix}0&0&1\\1&0&0\\0&0&0\end{bmatrix},\begin{bmatrix}0&1&0\\0&0&1\\0&0&0\end{bmatrix},\begin{bmatrix}0&0&1\\0&1&0\\0&1&0\end{bmatrix}\right\rangle .$$

- $C_U=\{0\}$ if $\dim U\geq 2$ and $C_U=C$ if $U=\{0\}$.
- If $\dim U=1$ then $|C_U|=2,2,2,2,4,4,4$.

$$b_0=\tfrac{13}{7}, b_1=2^4-1, b_2=2^7-1, b_3=2^{10}-1,\ldots, b_r=2^{4+3(r-1)}-1,\ldots$$

$$\begin{aligned} W(x,y) &= x^3+\begin{bmatrix}3\\2\end{bmatrix}(x-y)b_0+\begin{bmatrix}3\\3\end{bmatrix}b_1=x^3+7(x-y)y^2\frac{13}{7}+15y^3\\ &= x^3+13xy^2+2y^3. \end{aligned}$$

Zeta Functions for Rank Metric Codes

Definition 28 (The Zeta Function of C)

$$Z(T) := (q^m - 1)^{-1} \sum_{r \geq 0} b_r T^r.$$

$$b_r - (q^m + 1) b_{r-1} + q^m b_{r-2} = 0, r \notin \{0, ..., n - d^{\perp} - d + 2\}. \tag{1}$$

Definition 29 (The Zeta Polynomial of C)

$$P(T) := \sum_{r=0}^{n-d+1} p_r T^r,$$

$$p_r := (q^m - 1)^{-1} (b_r - (q^m + 1) b_{r-1} + q^m b_{r-2}).$$

The recurrence relation (1) yields

$$Z(T) = \frac{P(T)}{(1 - T)(1 - q^m T)}.$$

Zeta Functions, Zeta Polynomials, Weight Enumerators

$$\phi_{n,n-i}(x,y) := (q^m-1)^{-1}\left(M_{m\times n,i} - (q^m+1)M_{m\times n,i+1} + q^m M_{m\times n,i+2}\right).$$

$$\phi_n(T) := \sum_{r=0}^{n} \phi_{n,r}(x,y)T^r.$$

Theorem 30 (B., Blanco-Chacón, Duursma, Sheekey, 2017)

$$Z(T)\phi_n(T) = \frac{P(T)\phi_n(T)}{(1-T)(1-q^mT)} = \cdots + \frac{W(x,y)-x^n}{q^m-1}T^{n-d} + \cdots.$$

$P(T)$ is the unique polynomial of degree at most $n-d+1$ such that

$$W(x,y) = \sum_{i=0}^{n-d} p_i M_{m\times n,d+i}(x,y) + p_{n-d+1}x^n.$$

Example 31

For the $\mathbb{F}_q$-$[3\times 3,4,2]$ code C with $W(x,y)=x^3+13xy^2+2y^3$,

$$\begin{aligned}\phi_3(T) &= y^3+7(x-y)y^2T+\cdots\\ Z(T) &= \frac{13}{49}+\frac{15}{7}T+\frac{127}{7}T^2+\frac{1023}{7}T^3+\cdots,\\ P(T) &= \frac{13}{49}-\frac{12}{49}T+\frac{48}{49}T^2.\end{aligned}$$

Then

$$\begin{aligned}\frac{P(T)\phi_3(T)}{(1-T)(1-2^3T)} &= \frac{(13-12T+48T^2)(y^3+7(x-y)y^2T+\cdots)}{49(1-T)(1-8T)}\\ &= \cdots+\frac{1}{7}(13xy^2+2y^3)+\cdots\end{aligned}$$

and

$$\begin{aligned}p_0M_{3\times 3,2}+p_1M_{3\times 3,3}+p_2x^3 &= \frac{13}{49}(x^3+49xy^2+14y^3)-\frac{12}{49}(x^3+7y^3)+\frac{48}{49}x^3\\ &= x^3+13xy^2+2y^3.\end{aligned}$$

Invariance of the Zeta Polynomial

The weight enumerator of punctured/shortened MRD code is determined (it is MRD), so:

Theorem 32 (B., Blanco-Chacón, Duursma, Sheekey, 2017)

The zeta polynomial $P_C(T)$ is invariant under shortening and puncturing.

$$W(x,y) = \sum_{i=0}^{n-d} p_i M_{m\times n, d+i}(x,y) + p_{n-d+1}x^n.$$

$$\downarrow$$

puncturing

$$\downarrow$$

$$\sum_{i=0}^{n-d} p_i M_{m\times(n-1), d-1+i}(x,y) + p_{n-d+1}x^{n-1}.$$

Invariance of the Zeta Polynomial

The weight enumerator of punctured/shortened MRD code is determined (it is MRD), so:

Theorem 33 (B., Blanco-Chacón, Duursma, Sheekey, 2017)

The zeta polynomial $P(T)$ is invariant under shortening and puncturing.

$$W(x,y) = \sum_{i=0}^{n-d} p_i M_{m\times n, d+i}(x,y) + p_{n-d+1}x^n.$$

$$\downarrow$$

shortening

$$\downarrow$$

$$\sum_{i=0}^{n-1-d} p_i M_{m\times(n-1), d+i}(x,y) + p_{n-d}x^{n-1}.$$

Puncturing/Shortening Operations on Rank Weight Enumerators

- The weight enumerator of a punctured/shortened code depends on H.
- The **average** weight enum. after puncturing/shortening **is** determined.
- The average punctured/shortened weight enumerator can by computed by applying q-**derivatives** to $W(x,y)$.

$$\mathbf{P} := \begin{bmatrix} n \\ 1 \end{bmatrix}^{-1} (D_{q,x} + D_y) \text{ and } \mathbf{S} := \begin{bmatrix} n \\ 1 \end{bmatrix}^{-1} D_x.$$

Theorem 34 (B., Blanco-Chacón, Duursma, Sheekey, 2017)

1. $\mathbf{P}(W(x,y)) = \begin{bmatrix} n \\ 1 \end{bmatrix}^{-1} \sum_{\dim H = n-1} W_{\Pi_{H(C)}}(x,y),$
2. $\mathbf{S}(W(x,y)) = \begin{bmatrix} n \\ 1 \end{bmatrix}^{-1} \frac{1}{q^{n-1}} \sum_{\dim H = n-1, \langle h \rangle \not\subset H} W_{\Sigma_{h,H}}(x,y).$

Normalized Weight Enumerators and the Zeta Function

Arguments based on **puncturing/shortening** show that:

Theorem 35 (Duursma, 2001)

Let C be an $\mathbb{F}_q$-$[n, k, d]$ Hamming metric code. Then

$$\frac{P(T)}{(1-T)(1-qT)}(1-T)^{d+1} \equiv \mathscr{W}\left(\frac{1}{1-T}\right) \mod T^{n-d+1},$$

where

$$\mathscr{W}(T) := \frac{1}{q-1}\sum_{i=d}^{n}\binom{n}{i}^{-1} W_i T^{i-d},$$

is the **normalized weight enumerator** *of C.*

Gives a nice classification of **random divisible** self-dual codes wrt their $P(T)$.

We do not yet have a q-analogue of this result.

Normalized Weight Enumerators and the Zeta Function

Lemma 36 (Duursma, 2001)

Let $\mathscr{W}(T)$ be a n.w.e. Let $\mathscr{W}^{\mathbf{P}}(T)$ and $\mathscr{W}^{\mathbf{S}}(T)$ be the punctured and shortened n.w.e.s.

- $\mathscr{W}^{\mathbf{S}}(T) \equiv \mathscr{W}(T) \mod T^{n-d}$,
- $\mathscr{W}^{\mathbf{P}}(T) \equiv (1+T)\mathscr{W}(T) \mod T^{n-d+1}$.

$$
\begin{array}{rcl}
W(x,y) & = & p_0 M_{n,d}(x,y) + p_1 M_{n,d+1}(x,y) \\
 & = & (p_0\mathbf{P} + p_1\mathbf{S}) M_{n+1,d+1}(x,y) \\
\downarrow & & \quad\quad\downarrow \\
\mathscr{W}(T) & = & (p_0(1+T) + p_1 T)\mathscr{M}_{n+1,d+1}(T) \mod T^{n-d+1} \\
\Longrightarrow \mathscr{W}\left(\dfrac{T}{1-T}\right) & = & (p_0 + p_1 T)\dfrac{1}{1-T}\mathscr{M}_{n+1,d+1}\left(\dfrac{T}{1-T}\right) \mod T^{n-d+1}
\end{array}
$$

Normalized Weight Enumerators and the Zeta Function

Lemma 37 (Duursma, 2001)

Let $\mathscr{W}(T)$ be the Hamming distance n.w.e. Let $\mathscr{W}^{\mathbf{P}}(T)$ and $\mathscr{W}^{\mathbf{S}}(T)$ be the punctured and shortened n.w.e.s, resp.

- $\mathscr{W}^{\mathbf{S}}(T) \equiv \mathscr{W}(T) \mod T^{n-d}$,
- $\mathscr{W}^{\mathbf{P}}(T) \equiv (1+T)\mathscr{W}(T) \mod T^{n-d+1}$.

$$\begin{array}{rcl}
W(x,y) & = & p_0 M_{n,d}(x,y) + p_1 M_{n,d+1}(x,y) \\
& = & (p_0\mathbf{P} + p_1\mathbf{S}) M_{n+1,d+1}(x,y) \\
\downarrow & & \downarrow \\
\mathscr{W}(T) & = & (p_0(1+T) + p_1 T)\mathscr{M}_{n+1,d+1}(T) \mod T^{n-d+1} \\
\Longrightarrow \mathscr{W}\left(\frac{T}{1-T}\right) & = & P(T)\frac{1}{1-T}\mathscr{M}_{n+1,d+1}\left(\frac{T}{1-T}\right) \mod T^{n-d+1} \\
\Longrightarrow \mathscr{W}\left(\frac{T}{1-T}\right) & = & P(T)\frac{(1-T)^{d+1}}{(1-T)(1-qT)} \mod T^{n-d+1}
\end{array}$$

Invariance of Rank Normalized Weight Enumerators

Definition 38

The normalized weight enumerator (n.w.e.) of C is defined to be the polynomial,

$$\mathscr{W}(T) := (q^m - 1)^{-1} \sum_{i=d}^{n} \begin{bmatrix} n \\ i \end{bmatrix}^{-1} W_i T^{i-d}.$$

$\mathscr{W}^{\mathbf{P}}(T)$ and $\mathscr{W}^{\mathbf{S}}(T)$ are the n.w.e.s for $\mathbf{P}(W_C(x,y))$ and $\mathbf{S}(W_C(x,y))$.

Theorem 39 (B., Blanco-Chacón, Duursma, Sheekey, 2017)

- $\mathscr{W}^{\mathbf{S}}(T) \equiv \mathscr{W}(T) \mod T^{n-d}$,
- $\mathscr{W}^{\mathbf{P}}(T) \equiv (1 + q^d \alpha\varepsilon)\mathscr{W}(T) \mod T^{n-d+1}$.

where

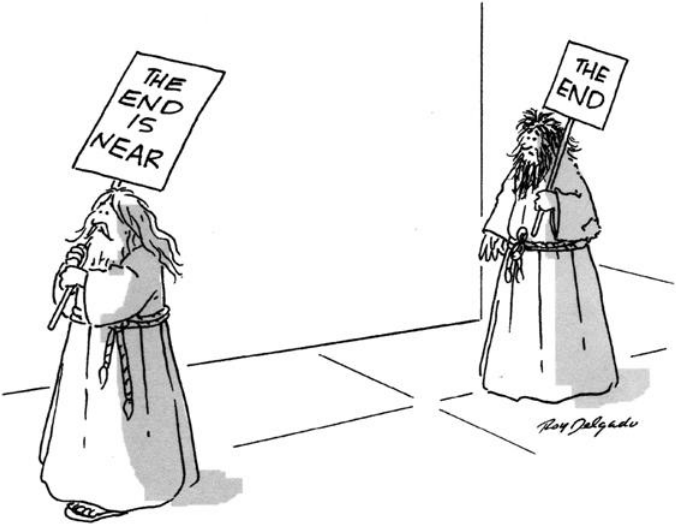
$$\alpha f(T) := Tf(T) \text{ and } \varepsilon f(T) := f(qT).$$

$$q\alpha\varepsilon = \varepsilon\alpha$$

Closing Remarks

- The theory of rank metric codes is still largely uncovered.
- The zeta polynomial can provide a tool for classifying codes with certain weight enumerators (e.g. divisible codes).
- The behaviours of zeroes of classes of codes is an interesting strand of research.
- q-commuting variables and q-derivatives feature in the theory of rank metric codes.
- Possible that many of the polynomial invariants of a rank metric code are best described in terms of q-commuting variables.

The End



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