Fundamental Operations on Rank Metric Codes

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What is Coding Theory About?

Coding Theory was introduced after Shannon’s noisy channel theorem (1948) for efficient communication across noisy channels.

sender transmits $c$, receiver gets $c$
What is Coding Theory About?

Coding Theory was introduced after Shannon’s noisy channel theorem (1948) for efficient communication across noisy channels.

sender transmits \( c \), receiver gets \( c + e = v \)
What is Coding Theory About?

Coding Theory was introduced after Shannon’s noisy channel theorem (1948) for efficient communication across noisy channels.

sender transmits $\mathbf{c}$, receivers want $\mathbf{c}$
Encoding

- \( m \in \mathbb{F}_q^k \) is a message
- encode \( m \) by multiplication with a full-rank \( k \times n \) matrix

\[
G : \mathbb{F}_q^k \longrightarrow \mathbb{F}_q^n : m \mapsto c = mG
\]

\[
C = \{ mG : m \in \mathbb{F}_q^k \}
\]
is an \( \mathbb{F}_q \)-[\( n, k, d \)] code.

What is \( d \)?

\( d \) is the minimum distance between a pair of distinct codewords.

Want low \( n \), high \( k \), high \( d \).
The higher \( d \) is, the more robust the code is to noise (packing problem).
Sphere-Packing

![Sphere-Packing Diagram]

- 000
- 001
- 010
- 011
- 100
- 101
- 110
- 111

E. Byrne
Fundamentals in Coding Theory

- Operations on codes - making new codes from old:
  - puncturing,
  - shortening,
  - extending,
  - concatenating,
  - products,
- Parameters of codes
  - length,
  - dimension,
  - minimum distance,
  - packing radius,
  - covering radius,
  - weight enumerators.
- Weight enumerators
  - MacWilliams duality theorem,
  - the zeta function.
### $q$-Analogues

<table>
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<tr>
<th>Subsets ${s_1, \ldots, s_k}$ of $[n]$</th>
<th>Subspaces $\langle s_1, \ldots, s_k \rangle$ of $\mathbb{F}_q^n$</th>
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<tbody>
<tr>
<td>Set cardinality</td>
<td>Vector space dimension</td>
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<tr>
<td>Binomial coefficients $\binom{n}{k}$</td>
<td>Gaussian coefficients $\binom{n}{k}_q$</td>
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<tr>
<td>Hamming weight of $(v_1, \ldots, v_n) \in \mathbb{F}_q^n$</td>
<td>$\mathbb{F}_q$-dimension of $\langle v_1, \ldots, v_n \rangle \subset \mathbb{F}_q^n$</td>
</tr>
<tr>
<td>Hamming weight of $(v_1, \ldots, v_n) \in \mathbb{F}_q^n$</td>
<td>$\mathbb{F}<em>q$-rank of $\begin{pmatrix} v</em>{11} &amp; v_{12} &amp; \cdots &amp; v_{1n} \ v_{21} &amp; v_{22} &amp; \cdots &amp; v_{2n} \ \vdots &amp; \vdots &amp; \vdots &amp; \vdots \ v_{m1} &amp; v_{m2} &amp; \cdots &amp; v_{mn} \end{pmatrix} \in \mathbb{F}_q^{m \times n}$</td>
</tr>
</tbody>
</table>
Introduced by Delsarte (1978) as a $q$-analogue of coding theory.

Independently introduced by Gabidulin (1986) and Roth (1991) for array error correction.

Studied more after 2000 in the context of code-based-cryptosystems.

Since 2008, generated interest among algebraic coding theorists due to their applicability in network error correction.

Many open problems in coding theory: only since 2015 have we seen new optimal families of rank metric codes.
Hamming Metric Codes

**Definition 1**
A linear $\mathbb{F}_q[n,k,d]$ code $C$ is a $k$-dimensional subspace of $\mathbb{F}_q^n$ of minimum Hamming distance

$$d = \min \{ d_H(c, c') : c, c' \in C \}.$$

- $d_H((u_1, \ldots, u_n), (v_1, \ldots, v_n)) := |\{ i \in [n] : u_i \neq v_i \}|.$
- $C$ is optimal if $k$ attains the max. possible dimension for fixed $n, d$.

**Theorem 2 (Singleton Bound, 1964)**
If $C$ is an $[n, k, d]$ code then $k \leq n - d + 1$.

- Codes that meet the Singleton bound are called **maximum distance separable** (MDS).
- MDS code exist for $n \leq q + 1$ (via Reed-Solomon codes).
- Segre (1955) conjectured that if $k \leq q$, $q$ odd then $n \leq q + 1$. 
Definition 3

A linear $\mathbb{F}_q^{m \times n}$ rank-metric code $C$ is a $k$-dimensional subspace of $\mathbb{F}_q^{m \times n}$ of minimum rank distance

$$d = \min \{ \text{rk}(A - B) : A, B \in C \}.$$ 

- $\text{rk}$ is a distance function on $\mathbb{F}_q^{m \times n}$.
- $C$ is optimal if $k$ attains the max. possible dimension for fixed $m, n, d$.

Theorem 4 (Rank Singleton Bound, Delsarte 1978)

If $C$ is an $[m \times n, k, d]$ code with $n \leq m$ then $k \leq m(n - d + 1)$.

- Codes that meet the rank Singleton bound are called maximum rank distance codes (MRD).
- MRD codes exist for all $m, n, d$. 
A Construction of MDS Codes

The Reed-Solomon Codes form a class of MDS codes. Choose $\alpha_1, \ldots, \alpha_n$ distinct in $\mathbb{F}_q^\times$.

$$\text{RS}(n, k) := \{c_f = (f(\alpha_1), \ldots, f(\alpha_n)) : f \in \mathbb{F}_q[x], \deg(f) \leq k - 1\}$$

Any pair of distinct polynomials $f, g \in \mathbb{F}_q[x]$ of degree $\leq k - 1$ have at most $k - 2$ common roots so

$$d_H(c_f, c_g) \geq n - k + 1.$$ 

From the Singleton bound its minimum distance is $\leq n - k + 1$, so RS$(n, k)$ is MDS.

Remark: For a basis-free approach, identify RS$(n, k)$ with

$$\{f \in \mathbb{F}_q[x], \deg(f) \leq k - 1\}.$$
A Construction of MRD Codes

The Delsarte-Gabidulin Codes form a class of MRD codes (1978, 1984).

- Let $L_m := \{ f_0 x + f_1 x^q + \cdots + f_k x^{q^{m-1}} : f_i \in \mathbb{F}_{q^m} \}$ (linearized polynomials in $\mathbb{F}_{q^m}[x]$)
- Choose $\alpha_1, \ldots, \alpha_n \subset \mathbb{F}_{q^m}$, linearly independent over $\mathbb{F}_q$

$$G(m, n, k) := \{ c_f = (f(\alpha_1), \ldots, f(\alpha_n)) : f \in L_m, \deg(f) \leq q^{k-1} \}$$

If $f, g \in L_m$, $\deg f, \deg g \leq q^{k-1} \Rightarrow \dim(\ker f \cap \ker g) \leq k - 1 \Rightarrow d_{rk}(c_f, c_g) \geq n - k + 1$.

- $G(n, k)$ is MRD by rank Singleton bound.
- $M_{n\times n}(\mathbb{F}_q) \cong L_n$ as rings (multiplication modulo $x^{q^n} - 1$).

For a basis-free approach, define

$$G(m, k) := \{ f_0 x + f_1 x^q + \cdots + f_k x^{q^{k-1}} : f_i \in \mathbb{F}_{q^m} \}.$$
Delsarte-Gabidulin codes admit fast decoding.

Until 2015 they were the only known family of MRD codes.

The twisted Delsarte-Gabidulin codes were discovered by Sheekey in 2015.

\[ \mathcal{H}(n, k) := \{ f_0 x + f_1 x^q + \cdots + f_k x^{q^{k-1}} + f_0 \theta^{q^h} x^{q^k} : f_i \in \mathbb{F}_{q^n} \} . \]

**Theorem 5**

If \( \theta \frac{q^n - 1}{q - 1} \neq (-1)^{nk} \) then \( \mathcal{H}(n, k) \) is MRD with parameters \([n \times n, kn, n - k + 1]\).

- The converse is false.
- The Delsarte-Gabidulin codes are \( \mathbb{F}_{q^n} \)-linear.
- The twisted Delsarte-Gabidulin codes are not always \( \mathbb{F}_{q^n} \)-linear.
- Few other families of rank-metric codes are known.
- Most MRD codes are not twisted Delsarte-Gabidulin codes.
The Hamming Weight Enumerator

The weight of a codeword is its distance to zero, wrt a given distance function. Given a linear code $C \subset \mathbb{F}_q^n$, its Hamming weight enumerator is

$$W(x, y) = \sum_{i=0}^{n} W_i x^{n-i} y^i,$$

where $W_t := |\{c \in C : d_H(c, 0) = t\}|$ for $0 \leq t \leq n$.

The dual $C^\perp := \{v \in \mathbb{F}_q^n : c \cdot v = 0 \ \forall c \in C\}$ has weight enumerator $W^\perp(x, y)$ st:

**Theorem 6 (MacWilliams Duality Theorem)**

$$W^\perp(x, y) = \frac{1}{|C|} W(x + (q - 1)y, x - y)$$
The Rank Weight Enumerator

Given a linear code \( C \subset \mathbb{F}_q^{m \times n} \), its rank weight enumerator is

\[
W(x, y) = \sum_{i=0}^{n} W_i x^{n-i} y^i,
\]

where \( W_t := |\{X \in C : \text{rk} X = t\}| \) for \( 0 \leq t \leq n \).

In 2008 Gadouleau and Yan derived the \( q \)-analogue of the MacWilliams duality theorem. (Also Delsarte 1970s via association schemes)

\[ C^\perp := \{Y \in \mathbb{F}_q^{m \times n} : \text{tr}(XY^T) = 0 \ \forall X \in C\} \] has weight enumerator \( W^\perp(x, y) \) st:

**Theorem 7 (Rank metric duality theorem)**

\[
W^\perp(x, y) = \frac{1}{|C|} \tilde{W}(x + (q^m - 1)y, x - y)
\]

where \( \tilde{W}(x, y) \) is a \( q \)-transform of \( W(x, y) \).
The weight enumerator is an important invariant of a code.

For example, weight enumerators relate codes to designs, strongly regular graphs and association schemes.

It also tells us precisely how effective the code is for transmitting information.

- For some extremal codes, the weight enumerator is determined.
- In the Hamming metric, this occurs for MDS codes.
- In the rank metric, this occurs for MRD codes.
- The MDS/MRD property of a weight enumerator is invariant under puncturing and shortening.
- The MDS/MRD weight enumerators are $\mathbb{Q}$-bases for the spaces of Hamming/rank metric weight enumerators.
Puncturing Hamming Metric Codes

Puncture an \( [n, k, d] \) code in \( \mathbb{F}_q^n \) by deleting the same coord. from each codeword. If \( d > 1 \) this results in an \( [n-1, k, \geq d-1] \) code.

**Example 8**

Puncture an \( \mathbb{F}_2-[8,4,4] \) code on the last coordinate to get an \( \mathbb{F}_2-[7,4,3] \) code.

\[
\begin{array}{cccccc}
00000000 & 11111111 & 0000000 & 1111111 \\
11100001 & 00011110 & 1110000 & 0001111 \\
10011001 & 01100110 & 1001100 & 0110011 \\
10000111 & 01111000 & 1000011 & 0111100 \\
01010101 & 10101010 & 0101010 & 1010101 \\
01001011 & 10110100 & 0100101 & 1011010 \\
00110011 & 11001100 & 0011001 & 1100110 \\
00101101 & 11010010 & 0010110 & 1101001 \\
\end{array}
\]

→

\[
\begin{array}{cccccc}
0000000 & 1111111 & 0000000 & 1111111 \\
1110000 & 0001111 & 1110000 & 0001111 \\
1001100 & 0110011 & 1001100 & 0110011 \\
1000011 & 0111100 & 1000011 & 0111100 \\
0101010 & 1010101 & 0101010 & 1010101 \\
0100101 & 1011010 & 0100101 & 1011010 \\
0011001 & 1100110 & 0011001 & 1100110 \\
0010110 & 1101001 & 0010110 & 1101001 \\
\end{array}
\]

The punctured code has a better rate, but worse minimum distance.
Shortening Hamming Metric Codes

Shorten an \([n, k, d]\) code by choosing the subcode with zero entries in a given coordinate and then deleting that same coordinate from each selected codeword. If \(d > 1\) this results in an \([n - 1, k - 1, \geq d]\) code.

**Example 9**

Shorten an \(\mathbb{F}_2-[8, 4, 4]\) code on the last coordinate to get an \(\mathbb{F}_2-[7, 3, 4]\) code.

\[
\begin{array}{ccc}
00000000 & 11111111 & 0000000 \\
11100001 & 00011110 & 0001111 \\
10011001 & 01100110 & 0110011 \\
10000111 & 01111000 & 0111100 \\
01010101 & 10101010 & \rightarrow \\
01001011 & 10110100 & 1011010 \\
00110011 & 11001100 & 1100110 \\
00101101 & 11010010 & 1101001 \\
\end{array}
\]

The shortened code has a worse rate, but may have a higher minimum distance.
We define shortening/puncturing as projections to $\mathbb{F}_q^{m \times (n-1)}$.

**Definition 10**

Let $H \in \mathbb{F}_q^{n \times (n-1)}$ have rank $n-1$. Let $h \in \mathbb{F}_q^n \setminus \text{col}(H)$.

The punctured and shortened codes of $C$ wrt $H$ are:

\[
\Pi_H(C) := \{XH : X \in C\} \subset \mathbb{F}_q^{m \times (n-1)} \text{ (punctured code)},
\]

\[
\Sigma_{h,H}(C) := \{XH : X \in C, Xh^T = 0\} \subset \mathbb{F}_q^{m \times (n-1)} \text{ (shortened code)}.
\]

**Example 11**

Let $E_i = [e_j^T : j \neq i]$, $e_j = [0,\ldots,1,\ldots,0]$.

- $\Pi_{E_i}(C)$ : delete the $i$th col of each elt of $C$.
- $\Sigma_{e_i,E_i}(C)$: delete the $i$th col of each elt of $C$ whose $i$th col is zero.
Example 12

Here’s an $\mathbb{F}_2$-$[4 \times 4, 3, 4]$ code, $C$.

\[
\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 \\
1 & 1 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 \\
1 & 0 & 0 & 1
\end{bmatrix}
\]

Then $\Sigma_{e_1,E_1}(C) = \{0\}$ as $C$ has weight enumerator $x^4 + 7xy^3$.

$Xh^T \neq 0$ for any $h \neq 0$, so all shortened codes of $C$ are trivial.
Example 13

Here's an $\mathbb{F}_2$-[3 × 3, 4, 2] code with $W(x, y) = x^3 + 13xy^2 + 2y^3$.

$$C = \langle \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \rangle.$$

<table>
<thead>
<tr>
<th>$H$</th>
<th>$h$</th>
<th>$W_{\Sigma_{h,H}}(x, y)$</th>
<th>$H$</th>
<th>$h$</th>
<th>$W_{\Sigma_{h,H}}(x, y)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E_2$</td>
<td>110</td>
<td>$x^2 + y^2$</td>
<td>$E_3$</td>
<td>001</td>
<td>$x^2 + 3y^2$</td>
</tr>
<tr>
<td>010</td>
<td>$x^2 + y^2$</td>
<td>111</td>
<td>$x^2 + y^2$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>111</td>
<td>$x^2 + y^2$</td>
<td>011</td>
<td>$x^2 + y^2$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>011</td>
<td>$x^2 + y^2$</td>
<td>101</td>
<td>$x^2 + 3y^2$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Duality of Puncturing and Shortening in the Hamming Metric

- The MDS property \((k = n - d + 1)\) is invariant under shortening and puncturing.
- Puncturing is really a projection to \(\mathbb{F}_q^{n-1}\).
- Shortening is projection of a subcode to \(\mathbb{F}_q^{n-1}\).
- We can puncture/shorten on several coords.

Recall \(C^\perp := \{x \in \mathbb{F}_q^n : c \cdot x = 0 \ \forall c \in C\}\).

**Theorem 14 (Duality of Puncturing and Shortening)**

Let \(P_i(C)\) and \(S_i(C)\) be the punctured and shortened codes of \(C\) at the \(i\)th coordinate, respectively. Then

\[
P_i(C)^\perp = S_i(C^\perp).
\]

**Pf:** (Easy) \(S_i(C^\perp) \subset P_i(C)^\perp\). Show equality by comparing dimensions.
Duality of Puncturing and Shortening in the Rank Metric

Recall for an $\mathbb{F}_q$-$[m \times n, k, d]$ code $C$,

$$C^\perp := \{ N \in \mathbb{F}_q^{m \times n} : \text{Tr}(MN^t) = 0 \text{ for all } M \in C \} \subseteq \mathbb{F}_q^{m \times n}.$$

**Theorem 15 (B., Ravagnani 2016)**

Duality of puncturing and shortening also holds for rank metric codes. In particular,

$$\Pi_{E_i}(C)^\perp = \Sigma_{e_i \in E_i}(C^\perp).$$

- $k^\perp := \dim(C^\perp) = mn - k$
- $C^{\perp \perp} = C$
- If $C$ is not $\mathbb{F}_q^m$-linear, its duals under the trace inner product and the scalar inner product are different.
Lemma 16 (Hamming Metric)

Let $C$ be an $\mathbb{F}_q$-$[n, k, d]$ code.

1. $P_i(C)$ is $[n - 1, k, \geq d - 1]$
2. $S_i(C)$ is $[n - 1, k - 1, \geq d]$.
3. If $C$ is MDS then so is $P_i(C)$.
4. If $C$ is MDS then so is $S_i(C)$.

Lemma 17 (Rank Metric)

Let $C$ be an $\mathbb{F}_q$-$[m \times n, k, d]$ code. Let $H \in \mathbb{F}_q^{n \times (n-1)}$ have rank $n - 1$ with $h \notin \text{col}(H)$.

1. $\Pi_H(C)$ is $[m \times (n - 1), k, \geq d - 1]$
2. $\Sigma_{h,H}(C)$ is $[m \times (n - 1), \geq k - m, \geq d]$.
3. If $C$ is MRD then so is $\Pi_H(C)$.
4. If $C$ is MRD then so is $\Sigma_{h,H}(C)$.
The Zeta Function of a Curve

- $\mathcal{C}$ non-singular projective curve over $\mathbb{F}_q$,
- $N_k$ the number of $\mathbb{F}_{q^k}$-rational points of $\mathcal{C}$,

The zeta-function of $\mathcal{C}$ is

$$Z(\mathcal{C}, T) = \exp\left( \sum_{k \geq 1} \frac{N_k}{k} T^k \right).$$

**Theorem 18 (Weil, Dwork)**

The zeta function of any non-singular projective curve of genus $g$ can be expressed as

$$Z(\mathcal{C}, T) = \frac{P(T)}{(1-T)(1-qT)},$$

some $P(T) \in \mathbb{Q}[T]$ $\deg P(T) \leq 2g$. $|\omega| = q^{-1/2}$ for each root $\omega$ of $P(T)$. 
Zeta Functions for Hamming-Metric Codes

**Definition 19 (Duursma 1999)**

The **zeta polynomial** of a (Hamming metric) \( F_q - [n, k, d] \) code \( C \) is the unique polynomial \( P(T) \) of degree at most \( n - d + 1 \) such that

\[
\frac{P(T)}{(1 - T)(1 - qT)}(Tx + (1 - T)y)^n = \cdots + \frac{W(x, y) - x^n}{q - 1} T^{n-d} + \cdots.
\]

The quotient

\[
Z(T) := \frac{P(T)}{(1 - T)(1 - qT)}
\]

is called the **zeta function** of \( C \).

- The weight enumerator \( M_{n,d} \) of an \( F_q - [n, d] \) MDS code is determined.\[P(T) = \sum_{i=0}^{n-d+1} p_i T^i \quad \Rightarrow \quad W(x, y) = \sum_{i=0}^{n-d} p_i M_{n,d+i}(x, y) + p_{n-d+1} x^n.\]
- If \( C \) is MDS then \( P(T) = 1 \).
MRD Weight Enumerators

- If $C$ is MRD, then its weight enumerator is determined (Delsarte, 1978).

\[
M_{m \times n, d}(x, y) = x^n + \sum_{i=d}^{n} (q^{m(i-d+1)} - 1) \binom{n}{i} y^{n-i-1} \prod_{t=0}^{i-1} (x - q^t y).
\]

- The MRD weight enumerators

\[
\{ M_{m \times n, d}(x, y) : 0 \leq d \leq n \} \cup \{x^n\}
\]

are a $\mathbb{Q}$-basis for the space of all $m \times n$ ‘weight enumerators’ (homog. polys of degree $n$).

- Given any $[m \times n, k, d]$ code $C$, there exist unique coefficients $p_i \in \mathbb{Q}$ s.t. for some $r$,

\[
W(x, y) = p_0 M_{m \times n, d}(x, y) + \cdots + p_r M_{m \times n, d+r}(x, y).
\]

- The $p_0, \ldots, p_r$ turn out to coincide with the coefficients of the zeta polynomial of $C$. 
Theorem 20 (B., Blanco-Chacón, Duursma, Sheekey, 2017)

\[ Z(T)\phi_n(T) = \frac{P(T)\phi_n(T)}{(1-T)(1-q^mT)} = \cdots + \frac{W(x,y) - x^n}{q^m-1} T^{n-d} + \cdots. \]

*P(T)* is the unique polynomial of degree at most \( n - d + 1 \) such that

\[ W(x,y) = \sum_{i=0}^{n-d} p_i M_{m\times n,d+i}(x,y) + p_{n-d+1} x^n. \]

\[ \phi_{n,r} = \binom{n}{r} \prod_{j=0}^{r-1} (x - q^j y) y^{n-r}, \]

\[ \phi_n(T) := \sum_{r=0}^{n} \phi_{n,r}(x,y) T^r, \]

if \( C \) is MRD then \( P(T) = 1. \)
Zeta Functions

- $Z(C, T)$ is the generating function for the number of points on a curve.
- $Z(T)$ is the generating function of **binomial moments** of a code.
- The binomial moments measure the average size of the **shortened subcodes**.
- The property $|\omega| = q^{-1/2}$ for every root $\omega$ of the zeta polynomial is called the Riemann hypothesis (RH).
- Many infinite families of codes with extremal Hamming weight enumerators sat. RH.
- It is conjectured that a sufficient condition for RH of a formally self-dual Hamming metric code is that it has weight distribution close to a random code.

**Question 1**

Which families of rank-metric codes satisfy the Riemann hypothesis?

$(|\omega| = q^{-m/2}?)$
Example 21
Any MRD code satisfies RH - it has \( P(T) = 1! \)

Example 22
- Take a (Hamming metric) extended binary QR code in \( \mathbb{F}_2^{18} \).
- Puncture and shorten this code to get a code in \( \mathbb{F}_2^{16} \).
- Express each resulting word in \( \mathbb{F}_2^{16} \) as a \( 4 \times 4 \) matrix.

The binomial moments are

\[
b_0 = 0, \ b_1 = 0, \ b_2 = 3/5, \ b_3 = 15, \ b_4 = 255
\]

\[
P(T) = \frac{(1 + 8T + 16T^2)}{25} = \frac{(1 + 4T)^2}{25}.
\]

The zeroes \( T = -1/4 \) have absolute value \( (2^4)^{-1/2} = 1/\sqrt{16} \) and so satisfy RH.

The zeta polynomial is that of a maximal elliptic curve over \( \mathbb{F}_{16} \).
The Riemann Hypothesis for Rank Metric Codes

Figure: Complex zeroes for $P(T)$ of $\mathbb{F}_4^{9 \times 9}$ in $\mathbb{F}_4^{18 \times 18}$. 
Theorem 23 (B., Blanco-Chacón, Duursma, Sheekey, 2017)

Let $P(T)$ be the zeta polynomial of an $\mathbb{F}_q-[m \times n, k, d]$ rank metric code and let $\theta$ be the negative of the sum of its reciprocal roots. Then

$$d \leq \log_q [(\theta + q^m + 1)(q - 1) + 1] - 1.$$ 

Follows due to MacWilliam’s duality theorem for rank metric codes.
Definition 24 (The shortened subcode of $C$)

We define the **shortened subcode** of $C$ wrt $U \subseteq \mathbb{F}_q^n$ as:

$$C_U := \left\{ X \in C : X u^T = 0 \quad \forall u \in U \right\}.$$ 

Definition 25 (The Binomial Moments of $C$)

$$b_r = \begin{cases} 
\begin{bmatrix} n \\
\dim U 
\end{bmatrix}^{-1} \sum_{\dim U = n - d - r} (|C_U| - 1) & \text{if } 0 \leq r \leq n - d \\
0 & \text{if } r < 0 \\
q^{k - mu} - 1, \quad u = n - d - r & \text{if } r > n - d^\perp - d
\end{cases}$$

Theorem 26 (B., Blanco-Chacón, Duursma, Sheekey, 2017)

$W(x, y)$ is completely determined by the $b_i$. 

Shortened Subcodes, Binomial Moments and $W(x,y)$

Example 27

Here's an $\mathbb{F}_2-[3 \times 3, 4, 2]$ code with $W(x, y) = x^3 + 13xy^2 + 2y^3$ and $d^\perp = 1$.

$$C = \langle \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \rangle.$$

- $C_U = \{0\}$ if $\text{dim} U \geq 2$ and $C_U = C$ if $U = \{0\}$.
- If $\text{dim} U = 1$ then $|C_U| = 2, 2, 2, 4, 4, 4$.

$$b_0 = \frac{13}{7}, b_1 = 2^4 - 1, b_2 = 2^7 - 1, b_3 = 2^{10} - 1, \ldots, b_r = 2^{4+3(r-1)} - 1, \ldots$$

$$W(x, y) = x^3 + \begin{bmatrix} 3 \\ 2 \end{bmatrix} (x - y)b_0 + \begin{bmatrix} 3 \\ 3 \end{bmatrix} b_1 = x^3 + 7(x - y)y^2 \frac{13}{7} + 15y^3$$

$$= x^3 + 13xy^2 + 2y^3.$$
### Definition 28 (The Zeta Function of $C$)

$$Z(T) := (q^m - 1)^{-1} \sum_{r \geq 0} b_r T^r.$$  

$$b_r - (q^m + 1)b_{r-1} + q^m b_{r-2} = 0, \quad r \notin \{0, ..., n - d^\perp - d + 2\}. \quad (1)$$

### Definition 29 (The Zeta Polynomial of $C$)

$$P(T) := \sum_{r=0}^{n-d+1} p_r T^r,$$  

$$p_r := (q^m - 1)^{-1}(b_r - (q^m + 1)b_{r-1} + q^m b_{r-2}).$$

The recurrence relation (1) yields

$$Z(T) = \frac{P(T)}{(1 - T)(1 - q^m T)}.$$
\[
\phi_{n,n-i}(x,y) := (q^m - 1)^{-1} \left(M_{m\times n,i} - (q^m + 1)M_{m\times n,i+1} + q^m M_{m\times n,i+2}\right).
\]

\[
\phi_n(T) := \sum_{r=0}^{n} \phi_{n,r}(x,y) T^r.
\]

**Theorem 30 (B., Blanco-Chacón, Duursma, Sheekey, 2017)**

\[
Z(T)\phi_n(T) = \frac{P(T)\phi_n(T)}{(1 - T)(1 - q^m T)} = \cdots + \frac{W(x,y) - x^n}{q^m - 1} T^{n-d} + \cdots.
\]

*P(T)* is the unique polynomial of degree at most \(n - d + 1\) such that

\[
W(x,y) = \sum_{i=0}^{n-d} p_i M_{m\times n,d+i}(x,y) + p_{n-d+1} x^n.
\]
Example 31

For the $\mathbb{F}_q-[3 \times 3, 4, 2]$ code $C$ with $W(x, y) = x^3 + 13xy^2 + 2y^3$,

\[
\phi_3(T) = y^3 + 7(x - y)y^2 T + \cdots
\]

\[
Z(T) = \frac{13}{49} + \frac{15}{7} T + \frac{127}{7} T^2 + \frac{1023}{7} T^3 + \cdots,
\]

\[
P(T) = \frac{13}{49} - \frac{12}{49} T + \frac{48}{49} T^2.
\]

Then

\[
\frac{P(T)\phi_3(T)}{(1 - T)(1 - 2^3 T)} = \frac{(13 - 12 T + 48 T^2)(y^3 + 7(x - y)y^2 T + \cdots)}{49(1 - T)(1 - 8 T)}
\]

\[
= \cdots + \frac{1}{7}(13xy^2 + 2y^3) + \cdots
\]

and

\[
p_0M_{3\times3,2} + p_1M_{3\times3,3} + p_2x^3 = \frac{13}{49}(x^3 + 49xy^2 + 14y^3) - \frac{12}{49}(x^3 + 7y^3) + \frac{48}{49}x^3
\]

\[
= x^3 + 13xy^2 + 2y^3.
\]
Invariance of the Zeta Polynomial

The weight enumerator of punctured/shortened MRD code is determined (it is MRD), so:

**Theorem 32 (B., Blanco-Chacón, Duursma, Sheekey, 2017)**

The zeta polynomial $P_C(T)$ is invariant under shortening and puncturing.

\[
W(x, y) = \sum_{i=0}^{n-d} p_i M_{m \times n, d+i}(x, y) + p_{n-d+1} x^n.
\]

↓

puncturing

↓

\[
\sum_{i=0}^{n-d} p_i M_{m \times (n-1), d-1+i}(x, y) + p_{n-d+1} x^{n-1}.
\]
Invariance of the Zeta Polynomial

The weight enumerator of punctured/shortened MRD code is determined (it is MRD), so:

**Theorem 33 (B., Blanco-Chacón, Duursma, Sheekey, 2017)**

The zeta polynomial $P(T)$ is invariant under shortening and puncturing.

$$W(x, y) = \sum_{i=0}^{n-d} p_i M_{m \times n, d+i}(x, y) + p_{n-d+1}x^n.$$  

↓

shortening

↓

$$\sum_{i=0}^{n-1-d} p_i M_{m \times (n-1), d+i}(x, y) + p_{n-d}x^{n-1}.$$
The weight enumerator of a punctured/shortened code depends on $H$.

The average weight enum. after puncturing/shortening is determined.

The average punctured/shortened weight enumerator can by computed by applying $q$-derivatives to $W(x,y)$.

\[ \mathbf{P} := \begin{bmatrix} n \\ 1 \end{bmatrix}^{-1} (D_q x + D_y) \text{ and } \mathbf{S} := \begin{bmatrix} n \\ 1 \end{bmatrix}^{-1} D_x. \]

**Theorem 34 (B., Blanco-Chacón, Duursma, Sheekey, 2017)**

1. \[ \mathbf{P}(W(x,y)) = \begin{bmatrix} n \\ 1 \end{bmatrix}^{-1} \sum_{\dim H = n-1} W_{\Pi_H(C)}(x,y), \]
2. \[ \mathbf{S}(W(x,y)) = \begin{bmatrix} n \\ 1 \end{bmatrix}^{-1} \frac{1}{q^{n-1}} \sum_{\dim H = n-1, \langle h \rangle \not\subset H} W_{\Sigma_{h,H}}(x,y). \]
Arguments based on **puncturing/shortening** show that:

**Theorem 35 (Duursma, 2001)**

Let $C$ be an $\mathbb{F}_q$-[n,k,d] Hamming metric code. Then

$$
P(T) \frac{(1-T)^{d+1}}{(1-T)(1-qT)} \equiv \mathcal{W} \left( \frac{1}{1-T} \right) \mod T^{n-d+1},$$

where

$$
\mathcal{W}(T) := \frac{1}{q-1} \sum_{i=d}^{n} \binom{n}{i}^{-1} W_i T^{i-d},
$$

is the normalized weight enumerator of $C$.

Gives a nice classification of **random divisible** self-dual codes wrt their $P(T)$.

We do not yet have a $q$-analogue of this result.
Lemma 36 (Duursma, 2001)

Let \( \mathcal{W}(T) \) be a n.w.e. Let \( \mathcal{W}^P(T) \) and \( \mathcal{W}^S(T) \) be the punctured and shortened n.w.e.s.

- \( \mathcal{W}^S(T) \equiv \mathcal{W}(T) \mod T^{n-d} \),
- \( \mathcal{W}^P(T) \equiv (1 + T)\mathcal{W}(T) \mod T^{n-d+1} \).

\[
\begin{align*}
\mathcal{W}(x, y) &= p_0 M_{n,d}(x, y) + p_1 M_{n,d+1}(x, y) \\
&= (p_0 P + p_1 S) M_{n+1,d+1}(x, y) \\
&\downarrow \\
\mathcal{W}(T) &= (p_0 (1 + T) + p_1 T) M_{n+1,d+1}(T) \mod T^{n-d+1} \\
&\Rightarrow \mathcal{W}\left(\frac{T}{1-T}\right) = (p_0 + p_1 T) \frac{1}{1-T} M_{n+1,d+1}\left(\frac{T}{1-T}\right) \mod T^{n-d+1}
\end{align*}
\]
Lemma 37 (Duursma, 2001)

Let $\mathcal{W}(T)$ be the Hamming distance n.w.e. Let $\mathcal{W}^P(T)$ and $\mathcal{W}^S(T)$ be the punctured and shortened n.w.e.s, resp.

- $\mathcal{W}^S(T) \equiv \mathcal{W}(T) \mod T^{n-d}$,
- $\mathcal{W}^P(T) \equiv (1 + T)\mathcal{W}(T) \mod T^{n-d+1}$.

\[
\begin{align*}
\mathcal{W}(x, y) &= p_0 M_{n,d}(x, y) + p_1 M_{n,d+1}(x, y) \\
&= (p_0 P + p_1 S) M_{n+1,d+1}(x, y)
\end{align*}
\]

\[
\begin{align*}
\mathcal{W}(T) &= (p_0(1 + T) + p_1 T) M_{n+1,d+1}(T) \mod T^{n-d+1}
\end{align*}
\]

\[
\begin{align*}
\mathcal{W}\left(\frac{T}{1 - T}\right) &= P(T) \frac{1}{1 - T} M_{n+1,d+1}\left(\frac{T}{1 - T}\right) \mod T^{n-d+1}
\end{align*}
\]

\[
\begin{align*}
\mathcal{W}\left(\frac{T}{1 - T}\right) &= P(T) \frac{(1 - T)^{d+1}}{(1 - T)(1 - qT)} \mod T^{n-d+1}
\end{align*}
\]
Invariance of Rank Normalized Weight Enumerators

**Definition 38**

The normalized weight enumerator (n.w.e.) of $C$ is defined to be the polynomial,

$$
\mathcal{W}(T) := (q^m - 1)^{-1} \sum_{i=d}^{n} \left[ \begin{array}{c} n \\ i \end{array} \right]^{-1} W_i T^{i-d}.
$$

$\mathcal{W}_P(T)$ and $\mathcal{W}_S(T)$ are the n.w.e.s for $P(W_C(x, y))$ and $S(W_C(x, y))$.

**Theorem 39 (B., Blanco-Chacón, Duursma, Sheekey, 2017)**

\[ \mathcal{W}_S(T) \equiv \mathcal{W}(T) \mod T^{n-d}, \]

\[ \mathcal{W}_P(T) \equiv (1 + q^d \alpha \varepsilon) \mathcal{W}(T) \mod T^{n-d+1}. \]

where

$$
\alpha f(T) := Tf(T) \text{ and } \varepsilon f(T) := f(qT).
$$

$q \alpha \varepsilon = \varepsilon \alpha$
The theory of rank metric codes is still largely uncovered.

The zeta polynomial can provide a tool for classifying codes with certain weight enumerators (e.g. divisible codes).

The behaviours of zeroes of classes of codes is an interesting strand of research.

$q$-commuting variables and $q$-derivatives feature in the theory of rank metric codes.

Possible that many of the polynomial invariants of a rank metric code are best described in terms of $q$-commuting variables.
The End


