Constructions of k-regular maps using algebraic geometry

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Nice, June 12th, 2017

Images from wiki

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Joined by: Chris, son of Miller[*] 

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Manifolds

A long time ago,

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A long time ago, when a concept of a manifold was just emerging. . .

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Theorem (Bertini, Whitney embedding theorem)

Every manifold of dimension m can be embedded in $\mathbb{R}^{2m+1}$. In fact, even in $\mathbb{R}^{2m}$.

The theorem is optimal in the sense that for infinitely many values of m (powers of 2) there exist manifolds (real projective spaces) that cannot be embedded in $\mathbb{R}^{2m-1}$.

Embeddings

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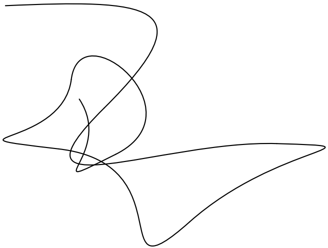

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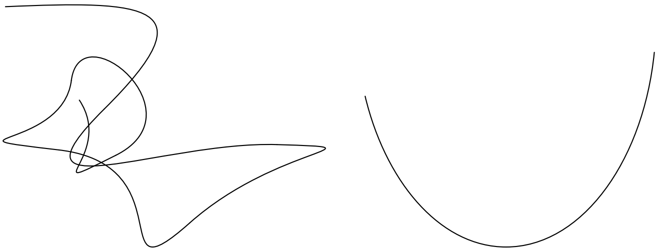

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For $k \in \mathbb{N}$ a continuous map $f: X \to \mathbb{R}^N$ is called (affinely) k-regular if for any distinct points $x_1, \ldots, x_k$ in X their images $f(x_1), \ldots, f(x_k)$ span a $k-1$ dimensional affine space.

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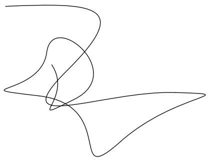
Definition (k-regular map)

For $k = 1$ a continuous map $f : X \to \mathbb{R}^N$ is called 1-regular if for any point x_1 in X its image $f(x_1)$ spans a 0 dimensional affine space (always true).



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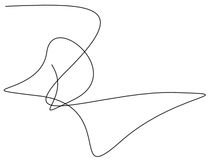
For $k = 2$ a continuous map $f : X \to \mathbb{R}^N$ is called 2-regular if for any distinct points x_1, x_2 in X their images $f(x_1), f(x_2)$ span a 1 dimensional affine space (a one-to-one map).



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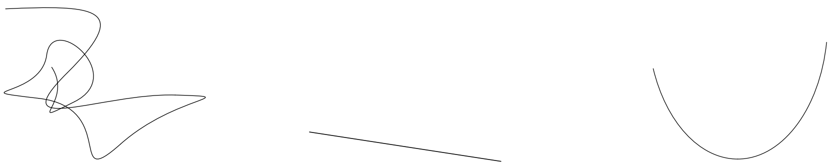
Definition (k-regular map)

For $k = 3$ a continuous map $f : X \to \mathbb{R}^N$ is called 3-regular if for any distinct points x_1, x_2, x_3 in X their images $f(x_1), f(x_2), f(x_3)$ span a 2 dimensional affine space (the first nontrivial case).



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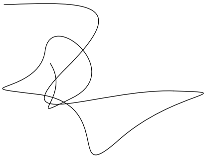

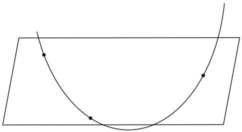

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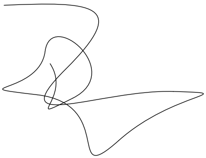

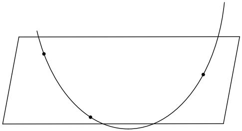

Fix X and k. What is the smallest possible N?

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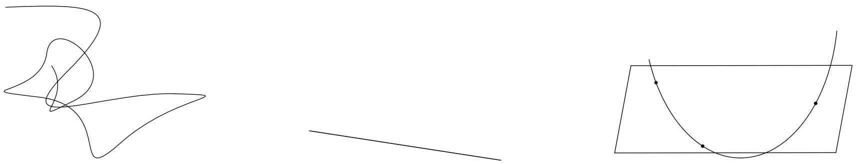
For $k \in \mathbb{N}$ a continuous map $f: X \to \mathbb{R}^N$ is called linearly k-regular if for any distinct points $x_1, \ldots, x_k$ in X their images $f(x_1), \ldots, f(x_k)$ span a k dimensional vector space.



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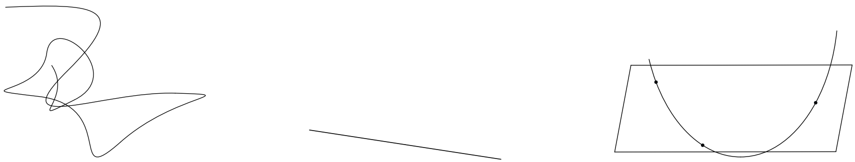
For $k \in \mathbb{N}$ a continuous map $f : X \to \mathbb{RP}^N$ is called projectively k-regular if for any distinct points $x_1, \ldots, x_k$ in X their images $f(x_1), \ldots, f(x_k)$ span a $k-1$ dimensional projective space.



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Interpolation

Suppose we have a bunch of points in X and prescribed values at these points. We want to find a continuous function $\varphi\colon X \to \mathbb{R}$ (with some extra restrictions) that takes the prescribed values at these points.

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Definition (k-interpolating space - Haar, Chebyshev)

A vector subspace Φ of the space of continuous functions $X \to \mathbb{R}$ is called k-interpolating if for any points $x_1, \ldots, x_k \in X$ and any values $\alpha_1, \ldots, \alpha_k \in \mathbb{R}$ there exists $\varphi \in \Phi$ such that $\varphi(x_i) = \alpha_i$ for all $1 \leq i \leq k$.

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How to find Φ of small dimension?

k-regular maps and interpolating subspaces

The following easy observation goes back at least to early sixties.

Fact

An N dimensional k-interpolating space for X exists if and only if there exists a linear k-regular map $X \to \mathbb{R}^N$.

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The problem of determining the smallest N (given X and k) attracted the attention of many algebraic topologists (and not only):
Blagojević, Bogatyi, Boltjanskii, Chisholm, F. Cohen, Handel, Lück, Ryskov, Saskin, Segal, Shekhtman, Vassiliev, Ziegler. . .

Setting

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1. One can (and indeed one does) ask similar questions changing the field from $\mathbb{R}$ to $\mathbb{C}$.
2. The problem is nontrivial (and maybe most interesting) even for the affine space $X = \mathbb{R}^m$ or $\mathbb{C}^m$!
3. The theorems that N cannot be too small were improved over the last 60 years with current world record holders: Blagojević, F. Cohen, Lück, Ziegler.

Monomial examples

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$t \mapsto (t, t^2, \ldots, t^{k-1})$
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Vandermonde determinant:

$$\det \begin{pmatrix} 1 & x_1 & x_1^2 & \dots & x_1^{k-1} \\ 1 & x_2 & x_2^2 & \dots & x_2^{k-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_k & x_k^2 & \dots & x_k^{k-1} \end{pmatrix} = \pm \prod_{1 \le i < j \le k} (x_i - x_j).$$

Monomial examples

$\mathbb{C}^1 \to \mathbb{C}^{k-1}$	$t \mapsto (t, t^2, \dots, t^{k-1})$ is k-regular
$\mathbb{C}^m \to \mathbb{C}^{2m}$	$(t_1, \dots, t_m) \mapsto (t_1, t_1^2, t_2, t_2^2, \dots, t_m, t_m^2)$ is 3-regular

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$\mathbb{C}^2 \to \mathbb{C}^6$	$(s, t) \mapsto (s, t, s^2, t^2, s^3, t^3)$

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$\mathbb{C}^2 \to \mathbb{C}^6$	$(s, t) \mapsto (s, t, s^2, t^2, s^3, t^3)$ is **not** 4-regular

The images of the points $(s_1, t_1), (s_1, t_2), (s_2, t_1), (s_2, t_2)$ are on an affine plane.

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$$\mathbb{C}^3 \to \mathbb{C}^9 \qquad (s,t,u) \mapsto (t,s,u,st,su,s^2-tu,t^2-s^3,u^2-t^3,u^3)$$

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Two constructions

Theorem (BJJM 2015)

If $k \geq 2$, then there exist k-regular maps $\mathbb{R}^m \to \mathbb{R}^{(m+1)(k-1)-1}$ and $\mathbb{C}^m \to \mathbb{C}^{(m+1)(k-1)-1}$.

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Bounds for k-regular maps $\mathbb{C}^m \to \mathbb{C}^N$ for small values of m

k	$m=1$	$m=2$ lower bound	$m=2$ upper bound	$m=3$ lower bound	$m=3$ upper bound
arbitrary	$k-1$	$2k-\alpha_2(k)-1$	$2k-2$	$3k-2\alpha_3(k)-1$	$4k-5$
2	1	2		3	
3	2	4		6	
4	3	6		7	9
5	4	8		12	
6	5	9	10	13	15
7	6	12		18	
8	7	14		18	21
9	8	15	16	24	
10	9	17	18	25	27
11	10	20	29	30	39
prime	$k-1$	$2k-2$		$3k-3$	$4k-5$

Bounds for any m, for k-regular maps $\mathbb{C}^m \to \mathbb{C}^N$

k	m arbitrary lower bound	upper bound
arbitrary	?	$(m+1)(k-1)-1$
2	m	
3	$2m$	
4	$2m$	$3m$
5	$4m$	
6	$4m$	$5m$
7	$6m$	
8	$6m$	$7m$
9	$6m$	$8m$
10	$6m$	$9m$
11	$10m$	$10m+9$
prime	$m(k-1)$	$(m+1)(k-1)-1$

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Let us start in a safe place we all know.

First k-regular maps

Recall our very first example (related to Vandermonde determinant):

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$$(t_1, \ldots, t_m) \mapsto (t_1, \ldots, t_m, t_1^2, t_1 t_2, \ldots, t_{i_1} t_{i_2} t_{i_3}, \ldots, t_m^{k-1})$$

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This would give $N = \binom{m+k-1}{m} - 1$, a result far from satisfactory.

Projections for embeddings

Let us go back to Whitney.

Theorem

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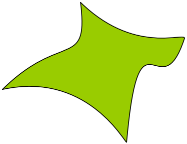

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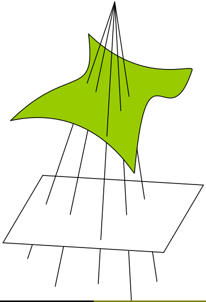
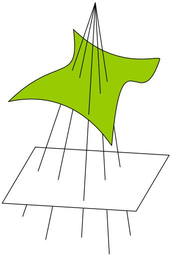


Projections and secant lines



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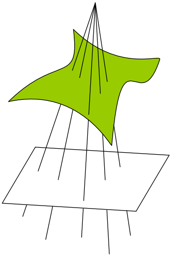

If the center of the projection does not intersect any secant line (or tangent space) to M, then the image is still an embedding of M.

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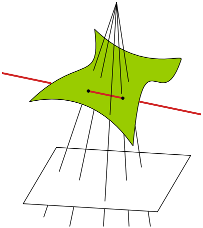

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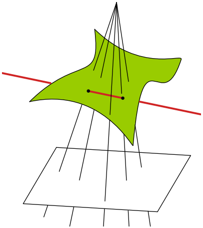
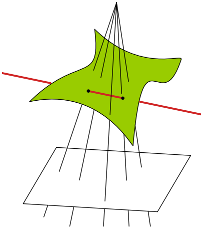


If the center of the projection does not intersect any secant line (or tangent space) to M, then the image is still an embedding of M. The dimension of the union of all secants and tangents $\sigma_2(M)$ is at most $2m+1$!

Projections and secant lines



If the center of the projection does not intersect any secant line (or tangent space) to M, then the image is still an embedding of M. The dimension of the union of all secants and tangents $\sigma_2(M)$ is at most $2m+1$! As long as $N > 2m+1$ we can project and still have a 2-regular map.

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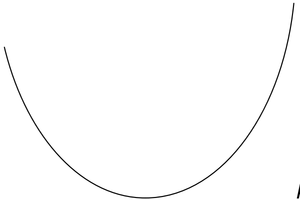

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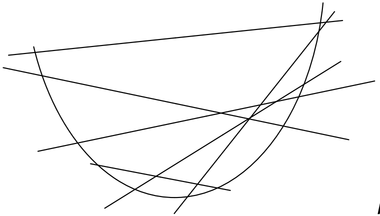

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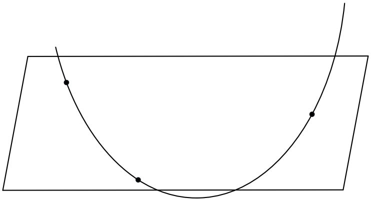

$k = 3$

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*Suppose $p \in \mathbb{C}^N$ is a point **not** in the k-th secant variety. Let $\pi\colon \mathbb{C}^N \to \mathbb{C}^{N-1}$ be the projection from p. Then the composition $\pi \circ f\colon \mathbb{C}^m \to \mathbb{C}^{N-1}$ is k-regular.*

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Local approach

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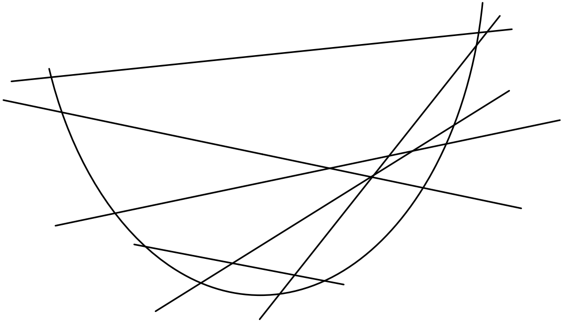
Local pictures

All secant lines to a parabola fill in whole $\mathbb{R}^2$.



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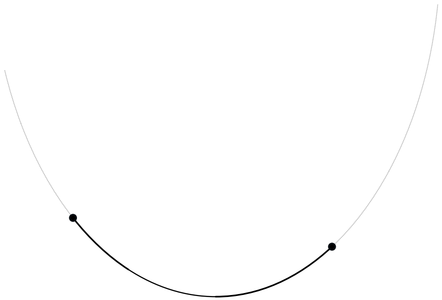

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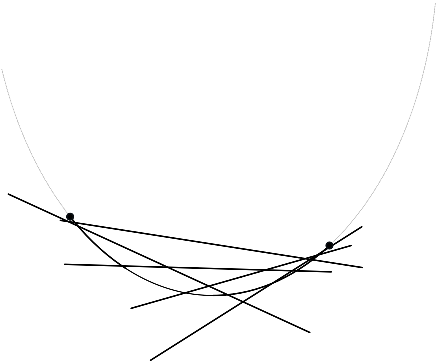

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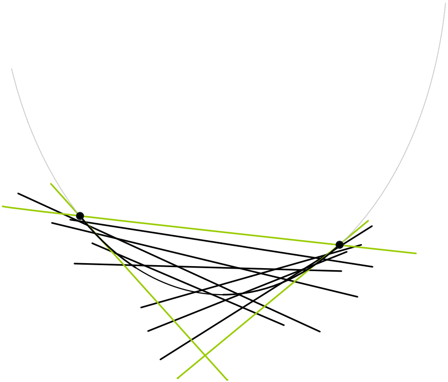

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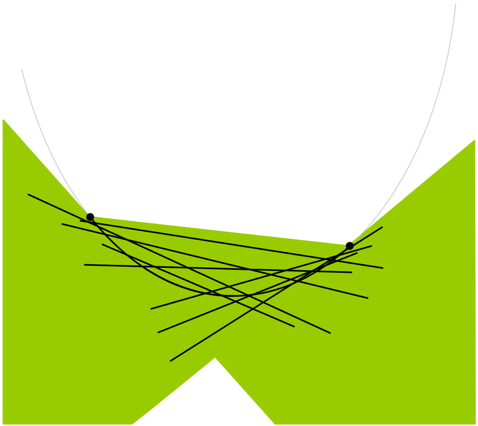

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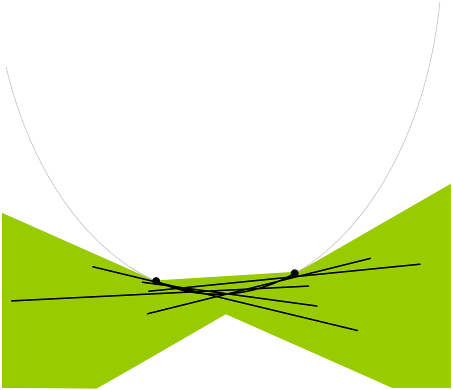

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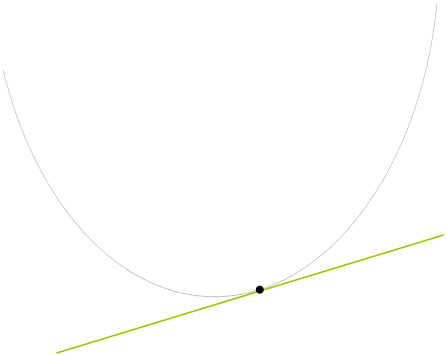

Punctual variant of a secant variety

Definition (Areole)

*Suppose $X \subset \mathbb{C}^N$ is an algebraic variety and $x \in X$ is a point. The k-**th areole** at x is*

$$a_k(X,x) := \overline{\left\{ \bigcup \langle R \rangle \mid R \in HilbP_{\leq k}(X,x), R \text{ is smoothable in } X \right\}}.$$

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In botany, *areoles* are the parts of the cactus stems out of which the spines grow. Remarkably, also beautiful cactus flowers grow out of areole, and analogously, regularity theorems spring out of the properties of the areole variety.

Hilbert scheme

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Here we are interested in a punctual version of the secant variety: we take only those subschemes from the smoothable component that are supported at a given point.

Why areoles?

Theorem

If a point $p \in \mathbb{C}^N$ avoids the areole at $x \in f(\mathbb{C}^m)$, then it also avoids secants from a small analytic neighborhood of x.

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We obtain existence of k-regular maps $\mathbb{C}^m \to \mathbb{C}^N$ for $N = (m+1)(k-1) - 1$.

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Finite local Gorenstein rings have a nice interpretation due to Macaulay, as quotients by an ideal given by derivatives annihilating a polynomial. For us the most important property is:

$$\langle R\rangle = \bigcup_{\substack{S\subset R \\ S \text{ is Gorenstein}}} \langle S\rangle,$$

where R is a finite scheme and S its Gorenstein subscheme.

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Although first option seems better in general, for $m \leq 3$ or $k < 14$ the second option is better, as all Gorenstein schemes are smoothable in this range.

More results

Through a detailed study of Gorenstein locus of the Hilbert scheme we obtain:

Theorem

If $2 \leq k \leq 9$ (or even 10) or $m \leq 2$, then there exist k-regular maps $\mathbb{R}^m \to \mathbb{R}^{m(k-1)}$ and $\mathbb{C}^m \to \mathbb{C}^{m(k-1)}$.

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I do not know, if it is 4-regular on $\mathbb{C}^3$.

Comparison

Theorem (Shekhtman, 2004)

The map

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Conjecture (BJJM, 2015)

There exists a continuous k-regular map $\mathbb{C}^m \to \mathbb{C}^N$ if and only if

$$N \geq m(k-1).$$

More interpolation

Definition (k-interpolating space)

A vector subspace Φ of the space of continuous complex valued functions $X \to \mathbb{C}$ is called k-interpolating if for any points $x_1, \ldots, x_k \in X$ and any values $\alpha_1, \ldots, \alpha_k \in \mathbb{C}$ there exists $\varphi \in \Phi$ such that $\varphi(x_i) = \alpha_i$ for all $1 \leq i \leq k$.

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Equivalent topological phrasing:

Definition (k-regular map)

A continuous map $g\colon X \to Gr(d, \mathbb{C}^N)$ is called Grassmann k-regular if for any distinct points $x_1, \ldots, x_k$ their images $g(x_1), \ldots, g(x_k)$ are k linear subspaces of $\mathbb{C}^N$ in a general linear positions, that is $\dim\langle g(x_1), \ldots, g(x_k)\rangle = kd$.

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$$g\colon X \to Gr(d, \mathbb{C}^{dN}),$$
$$g(x) = \langle (f(x), 0, \dots, 0), (0, f(x), 0, \dots, 0), \dots (0, \dots, 0, f(x)) \rangle.$$

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Corollary (Naive construction)

There exists a Grassmann k-regular map $\mathbb{C}^m \to Gr\left(d, \mathbb{C}^{d(m+1)(k-1)}\right)$ (or to $Gr\left(d, \mathbb{C}^{d(m(k-1)+1)}\right)$ for $k \leq 10$ or $m \leq 2$).

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Theorem (B, Januszkiewicz, Miller)

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Theorem (B, Januszkiewicz, Miller)

There exists a Grassmann k-regular map $\mathbb{C}^m \to \mathbb{C}^{(k-1)(m+d)}$.

Note that the Gorenstein method cannot work here!

Other types of k-regularity

We can also consider quadratic k-regularity, symplectic k-regularity, and other types of this notion.

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Morphisms, categorical approach to k-regularity... Unfortunately, so far we have no interesting theorems in this abstract approach.

Thank you!