

Automorphisms of \mathbb{P}^1 -bundles over rational surfaces

joint work with A. Fanelli and R. Terpereau

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Definitions and reminders

In this talk, the base-field is k , *algebraically closed field, of characteristic 0* (positive characteristic in preparation).

Aim

We want to study $\text{Aut}^\circ(X)$, when $X \rightarrow S$ is a \mathbb{P}^1 -bundle over a projective rational surface S .

Motivation

Study connected algebraic subgroups of the Cremona group $\text{Bir}(\mathbb{P}^3)$.

The works of Enriques (1893) and Umemura (1980-85) yield

connected algebraic subgroups of $\text{Bir}(\mathbb{P}^2)$ \leftrightarrow $\text{Aut}^\circ(X)$, X minimal rational surface ($X \simeq \mathbb{P}^2$ or $X \simeq \mathbb{F}_n$: \mathbb{P}^1 -bundles over \mathbb{P}^1).

connected algebraic subgroups of $\text{Bir}(\mathbb{P}^3)$ \leftrightarrow $\text{Aut}^\circ(X)$, X *some* minimal rational 3-fold (some of them are \mathbb{P}^1 -bundles over rational surface, e.g. over \mathbb{P}^2 or \mathbb{F}_n).

The result

Theorem

Let $\pi: X \rightarrow S$ be a \mathbb{P}^1 -bundle over a projective rational surface S . Then, there exists a surface S' , being either equal to \mathbb{P}^2 or a Hirzebruch surface \mathbb{F}_a , $a \geq 0$, a \mathbb{P}^1 -bundle $X' \rightarrow S'$, and a commutative diagram

$$\begin{array}{ccc} X & \overset{\psi}{\dashrightarrow} & X' \\ \pi \downarrow & & \pi' \downarrow \\ S & \xrightarrow{\eta} & S' \end{array}$$

such that ψ, η are $\text{Aut}^\circ(X)$ -equivariant birational maps (for some biregular actions of $\text{Aut}^\circ(X)$ on X', S, S') and such that the \mathbb{P}^1 -bundle $X' \rightarrow S'$ is of the following type:

- | | | | | |
|-----|-------------------|------------------------|--|----------------------------------|
| (1) | a decomposable | \mathbb{P}^1 -bundle | $\mathcal{F}_a^{b,c} \rightarrow \mathbb{F}_a$ | $a, b \geq 0, c \in \mathbb{Z};$ |
| (2) | a decomposable | \mathbb{P}^1 -bundle | $\mathcal{P}_b \rightarrow \mathbb{P}^2$ | $b \geq 0;$ |
| (3) | an Umemura | \mathbb{P}^1 -bundle | $\mathcal{U}_a^{b,c} \rightarrow \mathbb{F}_a$ | $a, b \geq 1, c \geq 2;$ |
| (4) | a Schwarzenberger | \mathbb{P}^1 -bundle | $\mathcal{S}_b \rightarrow \mathbb{P}^2$ | $b \geq 1.$ |

Presentations of \mathbb{P}^2 , \mathbb{F}_a

1 The *projective plane* is $\mathbb{P}^2 = (\mathbb{A}^3 \setminus \{0\})^2 / (\mathbb{G}_m)$

$$\begin{aligned} \mathbb{G}_m \times (\mathbb{A}^3 \setminus \{0\})^2 &\rightarrow (\mathbb{A}^3 \setminus \{0\})^2 \\ (\mu, (y_0, y_1, y_2)) &\mapsto (\mu y_0, \mu y_1, \mu y_2) \end{aligned}$$

The class of (y_0, y_1, y_2) is written $[y_0 : y_1 : y_2]$

2 Let $a \in \mathbb{Z}$. The *a -th Hirzebruch surface* is $\mathbb{F}_a = (\mathbb{A}^2 \setminus \{0\})^2 / (\mathbb{G}_m)^2$, where

$$\begin{aligned} (\mathbb{G}_m)^2 \times (\mathbb{A}^2 \setminus \{0\})^2 &\rightarrow (\mathbb{A}^2 \setminus \{0\})^2 \\ ((\mu, \rho), (y_0, y_1, z_0, z_1)) &\mapsto (\mu \rho^{-a} y_0, \mu y_1, \rho z_0, \rho z_1) \end{aligned}$$

The class of (y_0, y_1, z_0, z_1) will be written $[y_0 : y_1; z_0 : z_1]$. The projection

$$\tau_a: \mathbb{F}_a \rightarrow \mathbb{P}^1, \quad [y_0 : y_1; z_0 : z_1] \mapsto [z_0 : z_1]$$

identifies \mathbb{F}_a with $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1})$ as a \mathbb{P}^1 -bundle over \mathbb{P}^1 .

\mathbb{F}_a is isomorphic to \mathbb{F}_{-a} , via $[y_0 : y_1; z_0 : z_1] \mapsto [y_1 : y_0; z_0 : z_1]$

Decomposable \mathbb{P}^1 -bundles over \mathbb{P}^2

Let $b \in \mathbb{Z}$. We define $\mathcal{P}_b = (\mathbb{A}^2 \setminus \{0\}) \times (\mathbb{A}^3 \setminus \{0\}) / (\mathbb{G}_m)^2$ given by

$$\begin{aligned} (\mathbb{G}_m)^2 \times (\mathbb{A}^2 \setminus \{0\}) \times (\mathbb{A}^3 \setminus \{0\}) &\rightarrow (\mathbb{A}^2 \setminus \{0\}) \times (\mathbb{A}^3 \setminus \{0\}) \\ ((\mu, \rho), (y_0, y_1; z_0, z_1, z_2)) &\mapsto (\mu\rho^{-b}y_0, \mu y_1; \rho z_0, \rho z_1, \rho z_2) \end{aligned}$$

The class of $(y_0, y_1, z_0, z_1, z_2)$ will be written $[y_0 : y_1; z_0 : z_1 : z_2]$. The projection

$$\mathcal{P}_b \rightarrow \mathbb{P}^2, \quad [y_0 : y_1; z_0 : z_1 : z_2] \mapsto [z_0 : z_1 : z_2]$$

identifies \mathcal{P}_b with

$$\mathbb{P}(\mathcal{O}_{\mathbb{P}^2}(b) \oplus \mathcal{O}_{\mathbb{P}^2}) = \mathbb{P}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-b))$$

$$\mathcal{P}_b \xrightarrow{\cong} \mathcal{P}_{-b} \text{ via } [y_0 : y_1; z_0 : z_1 : z_2] \leftrightarrow [y_1 : y_0; z_0 : z_1 : z_2].$$

Decomposable \mathbb{P}^1 -bundles over \mathbb{F}_a

Let $a, b, c \in \mathbb{Z}$. We define $\mathcal{F}_a^{b,c} = (\mathbb{A}^2 \setminus \{0\})^3 / (\mathbb{G}_m)^3$ where

$$\begin{aligned} (\mathbb{G}_m)^3 \times (\mathbb{A}^2 \setminus \{0\})^3 &\rightarrow (\mathbb{A}^2 \setminus \{0\})^3 \\ ((\lambda, \mu, \rho), (x_0, x_1, y_0, y_1, z_0, z_1)) &\mapsto (\lambda\mu^{-b}x_0, \lambda\rho^{-c}x_1, \mu\rho^{-a}y_0, \mu y_1, \rho z_0, \rho z_1) \end{aligned}$$

The class of $(x_0, x_1, y_0, y_1, z_0, z_1)$ is written $[x_0 : x_1; y_0 : y_1; z_0 : z_1]$. The projection

$$\mathcal{F}_a^{b,c} \rightarrow \mathbb{F}_a, \quad [x_0 : x_1; y_0 : y_1; z_0 : z_1] \mapsto [y_0 : y_1; z_0 : z_1]$$

identifies $\mathcal{F}_a^{b,c}$ with

$$\mathbb{P}(\mathcal{O}_{\mathbb{F}_a}(bs_a) \oplus \mathcal{O}_{\mathbb{F}_a}(cf)) = \mathbb{P}(\mathcal{O}_{\mathbb{F}_a} \oplus \mathcal{O}_{\mathbb{F}_a}(-bs_a + cf))$$

as a \mathbb{P}^1 -bundle over \mathbb{F}_a , where $s_a, f \subset \mathbb{F}_a$ are given by $y_1 = 0$ and $z_1 = 0$.

$\mathcal{F}_a^{b,c} \xrightarrow{\simeq} \mathcal{F}_a^{-b,-c}$ via $x_0 \leftrightarrow x_1$.

Transition functions on \mathbb{F}_a

$$\mathbb{F}_a = \{[y_0 : y_1; z_0 : z_1] \mid [y_0 : y_1; z_0 : z_1] = [\mu\rho^{-a}y_0 : \mu y_1 : \rho z_0 : \rho z_1]\}$$

two open subsets $U_0, U_1 \subset \mathbb{F}_a$ (given by $z_0 \neq 0, z_1 \neq 0$) isomorphic to $\mathbb{P}^1 \times \mathbb{A}^1$:

$$\begin{aligned} \mathbb{P}^1 \times \mathbb{A}^1 &\xrightarrow{\cong} U_0 \subset \mathbb{F}_a \\ ([y_0 : y_1], z) &\mapsto [y_0 : y_1; 1 : z] \\ \mathbb{P}^1 \times \mathbb{A}^1 &\xrightarrow{\cong} U_1 \subset \mathbb{F}_a \\ ([y_0 : y_1], z) &\mapsto [y_0 : y_1; z : 1] \end{aligned}$$

glueing function:

$$([y_0 : y_1], z) \mapsto ([z^a y_0 : y_1], z)$$

Every \mathbb{P}^1 -bundle over \mathbb{P}^1 is isomorphic to \mathbb{F}_a for some $a \geq 0$:

Effective method: for each element $A \in \mathrm{GL}_2(k[z, \frac{1}{z}])$, there are $B \in \mathrm{GL}_2(k[\frac{1}{z}])$, $C \in \mathrm{GL}_2(k[z])$, such that

$$BAC = \begin{pmatrix} z^m & 0 \\ 0 & z^n \end{pmatrix} \rightsquigarrow a = |m - n|$$

Transition functions on $\mathcal{F}_a^{b,c}$

$$\mathcal{F}_a^{b,c} = \{[x_0 : x_1; y_0 : y_1; z_0 : z_1] = [\lambda\mu^{-b}x_0 : \lambda\rho^{-c}x_1 : \mu\rho^{-a}y_0 : \mu y_1 : \rho z_0 : \rho z_1]\}$$

$$\pi : \mathcal{F}_a^{b,c} \rightarrow \mathbb{F}_a, \quad [x_0 : x_1; y_0 : y_1; z_0 : z_1] \mapsto [y_0 : y_1; z_0 : z_1]$$

$$\mathbb{F}_a = \{[y_0 : y_1; z_0 : z_1] \mid [y_0 : y_1; z_0 : z_1] = [\mu\rho^{-a}y_0 : \mu y_1 : \rho z_0 : \rho z_1]\}$$

two open subsets $V_0 = \pi^{-1}(U_0)$, $V_1 = \pi^{-1}(U_1) \subset \mathcal{F}_a^{b,c}$ (given by $z_0 \neq 0$, $z_1 \neq 0$) isomorphic to $\mathbb{F}^b \times \mathbb{A}^1$:

$$\begin{array}{ccc} \mathbb{F}_b \times \mathbb{A}^1 & \xrightarrow{\cong} & V_0 \subset \mathcal{F}_a^{b,c} \\ ([x_0 : x_1; y_0 : y_1], z) & \mapsto & ([x_0 : x_1; y_0 : y_1; 1 : z]) \\ \mathbb{F}_b \times \mathbb{A}^1 & \xrightarrow{\cong} & V_1 \subset \mathcal{F}_a^{b,c} \\ ([x_0 : x_1; y_0 : y_1], z) & \mapsto & ([x_0 : x_1; y_0 : y_1; z : 1]) \end{array}$$

yields a \mathbb{F}_b -bundle $\mathcal{F}_a^{b,c} \rightarrow \mathbb{P}^1$, with a transition function

$$([x_0 : x_1; y_0 : y_1], z) \mapsto ([x_0 : x_1 z^c; y_0 z^a : y_1], \frac{1}{z}).$$

Not all \mathbb{P}^1 -bundles over \mathbb{F}_a are of this type!

Begining of the proof: descent lemma

Lemma (Descent lemma)

Let $\eta: \hat{S} \rightarrow S$ be a birational morphism between two smooth projective surfaces and let $\hat{\pi}: \hat{X} \rightarrow \hat{S}$ be a \mathbb{P}^1 -bundle. Then, there exists a \mathbb{P}^1 -bundle $\pi: X \rightarrow S$, unique up to isomorphism, and a birational map $\psi: \hat{X} \rightarrow X$ making the following diagram commutative. Moreover, the group $\text{Aut}^\circ(\hat{X})$ acts on the four varieties and the four rational maps are equivariant.

$$\begin{array}{ccc}
 \hat{X} & \xrightarrow{\psi} & X \\
 \hat{\pi} \downarrow & & \downarrow \pi \\
 \hat{S} & \xrightarrow{\eta} & S
 \end{array}$$

One can then reduce the study to \mathbb{P}^1 -bundles over \mathbb{P}^2 or Hirzebruch surfaces \mathbb{F}_n , $n \geq 0$ (and could avoid $n = 1$).

Lemma (Removal of jumping fibres)

Let $a \geq 0$, let $\pi: X \rightarrow \mathbb{F}_a$ be a \mathbb{P}^1 -bundle. There is an integer $b \geq 0$ and a dense open subset of $U \subset \mathbb{P}^1$ such that $(\tau_a\pi)^{-1}(p)$ is a Hirzebruch surface \mathbb{F}_b for each $p \in U$. Moreover, we have:

- 1 If $U = \mathbb{P}^1$, then $\tau_a\pi: X \rightarrow \mathbb{P}^1$ is a \mathbb{F}_b -bundle which is trivial on every affine open subset of \mathbb{P}^1 . In this case we say that η has no jumping fibre.
- 2 If one fibre $(\tau_a\pi)^{-1}(\{p\})$ is isomorphic to \mathbb{F}_c for some $c \neq b$, then $c - b$ is a positive even integer (we say that $\tau_a^{-1}(\{p\})$ is a jumping fibre), and the blow-up of the (unique) exceptional section of \mathbb{F}_c followed by the contraction of the strict transform of \mathbb{F}_c gives an $\text{Aut}^\circ(X)$ -equivariant birational map $X \dashrightarrow X'$ to another \mathbb{P}^1 -bundle over \mathbb{F}_a . After finitely many such steps, one gets case 1.

\rightsquigarrow Explicit algorithm, use transition matrices given by generic fibre being \mathbb{F}_n over $k(x)$, induction on determinant of matrices B, C

Proposition

Let $a \geq 0$ and let $\pi: X \rightarrow \mathbb{F}_a$ be a \mathbb{P}^1 -bundle.

1 For all integers $b \geq 0$, $c \in \mathbb{Z}$, the following are equivalent.

- 1** X is the gluing $Z_a^{b,c,P}$ of two copies of $\mathbb{F}_b \times \mathbb{A}^1$ along $\mathbb{F}_b \times \mathbb{A}^1 \setminus \{0\}$ by the automorphism $\nu_{c,P} \in \text{Aut}(\mathbb{F}_b \times \mathbb{A}^1 \setminus \{0\})$ given by $\nu_{c,P}: ([x_0 : x_1; y_0 : y_1], z) \mapsto ([x_0 : x_1 z^c + x_0 P(y_0, y_1, z); y_0 z^a : y_1], \frac{1}{z})$ for some $P \in k[y_0, y_1, z, \frac{1}{z}]$, homogeneous of degree b in y_0, y_1 , such that $\pi: X \rightarrow \mathbb{F}_a$ sends $([x_0 : x_1; y_0 : y_1], z) \in \mathbb{F}_b \times \mathbb{A}^1$ onto resp. $[y_0 : y_1; 1 : z] \in \mathbb{F}_a$ and $[y_0 : y_1; z : 1] \in \mathbb{F}_a$ on the 2 charts.
- 2** $\pi: X \rightarrow \mathbb{F}_a$ is the projectivisation of a rank two vector bundle \mathcal{E} which fits in a short exact sequence

$$0 \rightarrow \mathcal{O}_{\mathbb{F}_a} \rightarrow \mathcal{E} \rightarrow \mathcal{O}_{\mathbb{F}_a}(-bs_a + cf) \rightarrow 0$$

where $f, s_a \subset \mathbb{F}_a$ are given by $y_1 = 0$ and $z_1 = 0$.

- 2** If there exists $b \geq 0$ such that the preimage of each fibre of the \mathbb{P}^1 -bundle $\tau_a: \mathbb{F}_a \rightarrow \mathbb{P}^1$ is isomorphic to \mathbb{F}_b (no jumping fibre), then there is an integer $c \in \mathbb{Z}$ such that the above properties are satisfied. If $b > 0$, c is unique. If $b = 0$, then $\pi: X \rightarrow \mathbb{F}_a$ is decomposable.

The integers a, b, c are called *numerical invariants*.

Proposition

Let $a, b, c \in \mathbb{Z}$ with $a, b \geq 0$, such that $c \geq 0$ if $b = 0$. (*always possible*)

- 1 There is a unique isomorphism class of decomposable \mathbb{P}^1 -bundle $X \rightarrow \mathbb{F}_a$ with numerical invariants (a, b, c) : $\mathcal{F}_a^{b,c} \rightarrow \mathbb{F}_a$.
- 2 If $b = 0$ or $c \leq 1$ every \mathbb{P}^1 -bundle $X \rightarrow \mathbb{F}_a$ with numerical invariants (a, b, c) is decomposable, and thus isomorphic to $\mathcal{F}_a^{b,c} \rightarrow \mathbb{F}_a$.
- 3 If $b \geq 1$ and $c \geq 2$, every \mathbb{P}^1 -bundle with numerical invariants (a, b, c) is isomorphic to $Z_a^{b,c,P} \rightarrow \mathbb{F}_a$ (glueing), where

$$P(y_0, y_1, z) = \sum_{i=0}^b y_0^i y_1^{b-i} P_i(z) z^{ai+1}$$

and $P_i(z) \in k[z]_{\leq c-2-ai}$ (hence $P_i = 0$ if $c < ai + 2$), for $i = 0, \dots, b$.

The isomorphism class of the \mathbb{P}^1 -bundle is determined by the class of P , up to scalar multiplication by an element of k^* . The \mathbb{P}^1 -bundle is decomposable if and only if $P = 0$.

case $b \geq 1, c \geq 2$: two \mathbb{P}^1 -bundles $Z_a^{b,c,P} \rightarrow \mathbb{F}_a$ and $Z_a^{b,c',P'} \rightarrow \mathbb{F}_a$,

$$\begin{array}{ccc} \mathbb{F}_b \times \mathbb{A}^1 & \dashrightarrow & \mathbb{F}_b \times \mathbb{A}^1 \\ \mathbb{F}_b \times \mathbb{A}^1 \setminus \{0\} & \xrightarrow{\cong} & \mathbb{F}_b \times \mathbb{A}^1 \setminus \{0\} \\ \nu_{c,P}: ([x_0 : x_1; y_0 : y_1], z) & \mapsto & ([x_0 : x_1 z^c + x_0 P(y_0, y_1, z); y_0 z^a : y_1], \frac{1}{z}), \\ \nu_{c',P'}: ([x_0 : x_1; y_0 : y_1], z) & \mapsto & ([x_0 : x_1 z^{c'} + x_0 P'(y_0, y_1, z); y_0 z^a : y_1], \frac{1}{z}) \end{array}$$

\mathbb{P}^1 -bundles isomorphic $\Leftrightarrow \nu_{c',P'} = \alpha \nu_{c,P} \beta$ for some automorphisms α, β of the \mathbb{P}^1 -bundle $\mathbb{F}_b \times \mathbb{A}^1 \rightarrow \mathbb{P}^1 \times \mathbb{A}^1$. Since $b \geq 1$, α, β are of the form

$$\theta_{\lambda,Q}: \begin{array}{ccc} \mathbb{F}_b \times \mathbb{A}^1 & \xrightarrow{\cong} & \mathbb{F}_b \times \mathbb{A}^1 \\ ([x_0 : x_1; y_0 : y_1], z) & \mapsto & ([x_0 : \lambda x_1 + Q(y_0, y_1, z); y_0 : y_1], z) \end{array}$$

where $\lambda \in k^*$ and $Q \in k[y_0, y_1, z]_b$. The composition $\theta_{\lambda_2, Q_2} \nu_{c,P} \theta_{\lambda_1, Q_1}$ is

$$\begin{array}{ccc} \mathbb{F}_b \times \mathbb{A}^1 \setminus \{0\} & \xrightarrow{\cong} & \mathbb{F}_b \times \mathbb{A}^1 \setminus \{0\} \\ ([x_0 : x_1; y_0 : y_1], z) & \mapsto & ([x_0 : z^c \lambda_1 \lambda_2 x_1 + x_0 \tilde{P}(y_0, y_1, z); z^a y_0 : y_1], \frac{1}{z}), \\ \tilde{P}(y_0, y_1, z) & = & \lambda_2 P(y_0, y_1, z) + \lambda_2 Q_1(y_0, y_1, z) z^c + Q_2(y_0 z^a, y_1, \frac{1}{z}). \end{array}$$

To get a transition function of the form $\nu_{c',P'}$, we get $c' = c, \lambda_1 \lambda_2 = 1$.

Corollary

Let $a, b, c \in \mathbb{Z}$, with $a \geq 0$, $b \geq 1$ and $c \geq 2$.

The isomorphism classes of non-decomposable \mathbb{P}^1 -bundles $X \rightarrow \mathbb{F}_a$ with numerical invariants (a, b, c) are parametrised by the projective space

$$\mathcal{M}_a^{b,c} = \mathbb{P} \left(\bigoplus_{i=0}^b y_0^i y_1^{b-i} \cdot k[z]_{\leq c-2-ai} \right)$$

- 1 Explicit description of the action of $\text{Aut}(\mathbb{F}_a)$ on $\mathcal{M}_a^{b,c}$. *Action of GL_2 , preserves the direct sum!*
- 2 If $c - 2 - ai = 0$ for some i , then GL_2 preserves one typical element *Umemura bundles*
- 3 *If one fibre of $\mathbb{F}_a \rightarrow \mathbb{P}^1$ is invariant, we can reduce to decomposable case.*
- 4 If $a = 0$, one can reduce to the case where the diagonal $\text{PGL}_2 \subset \text{PGL}_2 \times \text{PGL}_2 = \text{Aut}(\mathbb{P}^1 \times \mathbb{P}^1)$ lifts. *Explicit family \hat{S}_b , with $c = b + 2$.*

Classification of \mathbb{P}^1 -bundles over Hirzebruch surfaces \mathbb{F}_a

Proposition

Let $a \geq 0$ and let $\pi: X \rightarrow \mathbb{F}_a$ be a \mathbb{P}^1 -bundle. Then, there exist $b, c \in \mathbb{Z}$ such that one of the following holds.

- 1 X is isomorphic to a decomposable \mathbb{P}^1 -bundle $\mathcal{F}_a^{b,c} \rightarrow \mathbb{F}_a$;
- 2 X is isomorphic to an Umemura \mathbb{P}^1 -bundle $\mathcal{U}_a^{b,c} \rightarrow \mathbb{F}_a$;
- 3 We have $a = 0$ and there is an isomorphism $\kappa: \mathbb{F}_0 \xrightarrow{\cong} \mathbb{F}_0$ such that $\kappa\pi$ is isomorphic to $\hat{S}_b \rightarrow \mathbb{F}_0$; or
- 4 There exists a \mathbb{P}^1 -bundle $\tau: \mathbb{F}_a \rightarrow \mathbb{P}^1$ and a closed point $p \in \mathbb{P}^1$ such that $(\tau\pi)^{-1}(p)$ is invariant by $\text{Aut}^\circ(X)$.

In cases 3-4, there is a decomposable bundle $\mathcal{F}_a^{b,c} \rightarrow \mathbb{F}_a$ and a commutative diagram

$$\begin{array}{ccc} X & \overset{\psi}{\dashrightarrow} & \mathcal{F}_a^{b,c} \\ & \searrow & \downarrow \\ & & \mathbb{F}_a \end{array}$$

where ψ is a birational map satisfying $\psi \text{Aut}^\circ(X) \psi^{-1} \subsetneq \text{Aut}^\circ(\mathcal{F}_a^{b,c})$.

The \mathbb{P}^1 -bundles $\mathcal{S}_b \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$ and $\mathcal{S}_b \rightarrow \mathbb{P}^2$ Definition (Schwarzenberger \mathbb{P}^1 -bundles)

Let $b \geq -1$ and let $\kappa: \mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \mathbb{P}^2$ be the $(2 : 1)$ -covering defined by

$$\begin{aligned} \kappa: \quad \mathbb{P}^1 \times \mathbb{P}^1 &\longrightarrow \mathbb{P}^2 \\ ([y_0 : y_1], [z_0 : z_1]) &\longmapsto [y_0 z_0 : y_0 z_1 + y_1 z_0 : y_1 z_1], \end{aligned}$$

branch locus: diagonal $\Delta \subset \mathbb{P}^1 \times \mathbb{P}^1$, ramification locus:

$\Gamma = \{[X : Y : Z] \mid Y^2 = 4XZ\} \subset \mathbb{P}^2$. The b -th Schwarzenberger \mathbb{P}^1 -bundle $\mathcal{S}_b \rightarrow \mathbb{P}^2$ is the \mathbb{P}^1 -bundle given by

$$\mathcal{S}_b = \mathbb{P}(\kappa_* \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(-b-1, 0)) \rightarrow \mathbb{P}^2.$$

$\mathcal{S}_{-1} \simeq \mathcal{P}_1 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2})$, $\mathcal{S}_0 \simeq \mathcal{P}_0 = \mathbb{P}^1 \times \mathbb{P}^2$, $\mathcal{S}_1 \simeq \mathbb{P}(T_{\mathbb{P}^2})$.

Proposition

For each $b \geq 1$, there is an involution $\iota \in \text{Aut}(\hat{\mathcal{S}}_b)$ such that $\mathcal{S}_b = \hat{\mathcal{S}}_b / \langle \iota \rangle$.

$$\begin{array}{ccc} \hat{\mathcal{S}}_b & \longrightarrow & \mathcal{S}_b \\ \downarrow & & \downarrow \\ \mathbb{P}^1 \times \mathbb{P}^1 & \xrightarrow{\quad \kappa \quad} & \mathbb{P}^2 \end{array}$$

The \mathbb{P}^1 -bundles over \mathbb{P}^2

Extensive literature. Here we can use the work made for Hirzebruch surfaces, and recover results explicitly.

Proposition

Let $X \rightarrow \mathbb{P}^2$ be a \mathbb{P}^1 -bundle, and let $H \subset \mathrm{PGL}_3$ be the image of $\mathrm{Aut}^\circ(X) \rightarrow \mathrm{Aut}(\mathbb{P}^2) = \mathrm{PGL}_3$. Then we have the following alternative.

- 1** $H = \mathrm{Aut}(\mathbb{P}^2)$ and X is decomposable or equal to $\mathcal{S}_1 \simeq \mathbb{P}(T_{\mathbb{P}^2})$.
- 2** H is conjugated to $\mathrm{Aut}(\mathbb{P}^2, C) \simeq \mathrm{PGL}_2$, where C is a smooth conic in \mathbb{P}^2 and $X \simeq \mathcal{S}_b$ for $b \geq 2$.
- 3** H fixes a point in \mathbb{P}^2 and there is a \mathbb{P}^1 -bundle $X' \rightarrow \mathbb{F}_1$ and a commutative diagram

$$\begin{array}{ccc}
 X' & \xrightarrow{\psi} & X \\
 \pi' \downarrow & & \pi \downarrow \\
 \mathbb{F}_1 & \xrightarrow{\eta} & \mathbb{P}^2
 \end{array}$$

where η is the contraction morphism and ψ is a birational morphism such that $\psi^{-1} \mathrm{Aut}^\circ(X) \psi \subset \mathrm{Aut}^\circ(X')$.

The result

Theorem

Let $\pi: X \rightarrow S$ be a \mathbb{P}^1 -bundle over a projective rational surface S . Then, there exists a surface S' , being either equal to \mathbb{P}^2 or a Hirzebruch surface \mathbb{F}_a , $a \geq 0$, a \mathbb{P}^1 -bundle $X' \rightarrow S'$, and a commutative diagram

$$\begin{array}{ccc} X & \overset{\psi}{\dashrightarrow} & X' \\ \pi \downarrow & & \pi' \downarrow \\ S & \xrightarrow{\eta} & S' \end{array}$$

such that ψ, η are $\text{Aut}^\circ(X)$ -equivariant birational maps (for some biregular actions of $\text{Aut}^\circ(X)$ on X', S, S') and such that the \mathbb{P}^1 -bundle $X' \rightarrow S'$ is of the following type:

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|-----|-------------------|------------------------|--|-----------------------------------|
| (1) | a decomposable | \mathbb{P}^1 -bundle | $\mathcal{F}_a^{b,c} \rightarrow \mathbb{F}_a$ | $a, b \geq 0, c \in \mathbb{Z}$; |
| (2) | a decomposable | \mathbb{P}^1 -bundle | $\mathcal{P}_b \rightarrow \mathbb{P}^2$ | $b \geq 0$; |
| (3) | an Umemura | \mathbb{P}^1 -bundle | $\mathcal{U}_a^{b,c} \rightarrow \mathbb{F}_a$ | $a, b \geq 1, c \geq 2$; |
| (4) | a Schwarzenberger | \mathbb{P}^1 -bundle | $\mathcal{S}_b \rightarrow \mathbb{P}^2$ | $b \geq 1$. |