Waring decompositions and identifiability via Bertini and Macaulay2 softwares

Elena Angelini

Dip. di Ingegneria dell'Informazione e Scienze Matematiche
Università degli Studi di Siena

MEGA 2017, Université Nice Sophia Antipolis
Nice (France), June 12 - 16, 2017

PRELIMINARIES

- Waring decomposition
- Waring identifiability
- Simultaneous Waring decomposition
- Real identifiability

RECENT RESULTS

- Main theorem
- Numerical Algebraic Geometry
- Proof of the main theorem
- Catalecticant intersections in non-identifiable cases

RELATED TOPIC

- Sub-generic rank cases: Hessian criterion

REFERENCES

Let $\mathbb{F}$ be either $\mathbb{C}$ or $\mathbb{R}$. Let $n, d \in \mathbb{N}$ and let $\mathbb{F}[x_0, \ldots, x_n]_d$ be the space of homogeneous polynomials of degree d in $n+1$ variables over $\mathbb{F}$.

Definition

A *Waring decomposition* of $p \in \mathbb{F}[x_0, \ldots, x_n]_d$ is given by linear forms $\ell_i \in \mathbb{F}[x_0, \ldots, x_n]_1$, $i \in \{1, \ldots, k\}$, s.t.

$$p = \sum_{i=1}^{k} \ell_i^d. \qquad (1)$$

The *(Waring) rank* of p over $\mathbb{F}$ is the minimal k appearing in (1).

Theorem (Alexander-Hirschowitz, 1995 [AH])

A general $p \in \mathbb{C}[x_0, \ldots, x_n]_d$ has rank $k = \left\lceil \binom{n+d}{d} \frac{1}{n+1} \right\rceil$, except for

- $(n, 2)$, $k = n + 1$;
- $(n, d) \in \{(2, 4), (3, 4), (4, 3), (4, 4)\}$, $k = \left\lceil \binom{n+d}{d} \frac{1}{n+1} \right\rceil + 1$.

Definition

The rank of the general form is called the *generic rank*.

Corollary

If $\binom{n+d}{d} \frac{1}{n+1} \in \mathbb{N}$, then the general $p \in \mathbb{C}[x_0, \ldots, x_n]_d$ has finitely many Waring decompositions, with some of the exceptions above.

Theorem (Alexander-Hirschowitz, 1995 [AH])

A general $p \in \mathbb{C}[x_0, \ldots, x_n]_d$ has rank $k = \left\lceil \binom{n+d}{d} \frac{1}{n+1} \right\rceil$, except for

- $(n, 2)$, $k = n + 1$;
- $(n, d) \in \{(2, 4), (3, 4), (4, 3), (4, 4)\}$, $k = \left\lceil \binom{n+d}{d} \frac{1}{n+1} \right\rceil + 1$.

Definition

The rank of the general form is called the *generic rank*.

Corollary

If $\binom{n+d}{d} \frac{1}{n+1} \in \mathbb{N}$, then the general $p \in \mathbb{C}[x_0, \ldots, x_n]_d$ has finitely many Waring decompositions, with some of the exceptions above.

Theorem (Alexander-Hirschowitz, 1995 [AH])

A general $p \in \mathbb{C}[x_0, \dots, x_n]_d$ has rank $k = \left\lceil \binom{n+d}{d} \frac{1}{n+1} \right\rceil$, except for

- $(n, 2)$, $k = n+1$;
- $(n, d) \in \{(2,4), (3,4), (4,3), (4,4)\}$, $k = \left\lceil \binom{n+d}{d} \frac{1}{n+1} \right\rceil + 1$.

Definition

The rank of the general form is called the *generic rank*.

Corollary

If $\binom{n+d}{d} \frac{1}{n+1} \in \mathbb{N}$, then the general $p \in \mathbb{C}[x_0, \dots, x_n]_d$ has finitely many Waring decompositions, with some of the exceptions above.

Definition

The general $p \in \mathbb{F}[x_0, \ldots, x_n]_d$ is *Waring identifiable* over $\mathbb{F}$ if the expression (1) is unique up to a permutation and scaling of the summands.

Applications: engineering [AGHKT], chemistry [AD],...

Theorem (Galuppi-Mella, 2016 [GM])

A general $p \in \mathbb{C}[x_0, \ldots, x_n]_d$ is Waring identifiable over $\mathbb{C}$ with rank k if and only if:

- $(n, d, k) = (1, 2t+1, t+1)$, [Sy];
- $(n, d, k) = (2, 5, 7)$, [Hi];
- $(n, d, k) = (3, 3, 5)$, Sylvester's Pentahedral Theorem [Sy].

Definition

The general $p \in \mathbb{F}[x_0, \ldots, x_n]_d$ is *Waring identifiable* over $\mathbb{F}$ if the expression (1) is unique up to a permutation and scaling of the summands.

Applications: engineering [AGHKT], chemistry [AD],...

Theorem (Galuppi-Mella, 2016 [GM])

A general $p \in \mathbb{C}[x_0, \ldots, x_n]_d$ is Waring identifiable over $\mathbb{C}$ with rank k if and only if:

- $(n, d, k) = (1, 2t + 1, t + 1)$, [Sy];
- $(n, d, k) = (2, 5, 7)$, [Hi];
- $(n, d, k) = (3, 3, 5)$, Sylvester's Pentahedral Theorem [Sy].

Definition

The general $p \in \mathbb{F}[x_0, \dots, x_n]_d$ is *Waring identifiable* over $\mathbb{F}$ if the expression (1) is unique up to a permutation and scaling of the summands.

Applications: engineering [AGHKT], chemistry [AD],...

Theorem (Galuppi-Mella, 2016 [GM])

A general $p \in \mathbb{C}[x_0, \dots, x_n]_d$ is Waring identifiable over $\mathbb{C}$ with rank k if and only if:

- $(n, d, k) = (1, 2t + 1, t + 1)$, [Sy];
- $(n, d, k) = (2, 5, 7)$, [Hi];
- $(n, d, k) = (3, 3, 5)$, Sylvester's Pentahedral Theorem [Sy].

Attempts to recover Waring identifiability when it fails over $\mathbb{C}$

- By introducing the *simultaneous* Waring decomposition of more forms with the same number of summands
- By requiring *real* identifiability

Let $n, r \in \mathbb{N}$ and let $d_1, \ldots, d_r \in \mathbb{N}$ such that $d_1 \leq \ldots \leq d_r$. A *polynomial vector* $f = (f_1, \ldots, f_r)$ is a vector of homogeneous polynomials $f_j \in \mathbb{F}[x_0, \ldots, x_n]_{d_j}$, $j \in \{1, \ldots, r\}$.

Definition

A *(simultaneous) Waring decomposition* of $f = (f_1, \ldots, f_r)$ is given by $\ell_1, \ldots, \ell_k \in \mathbb{F}[x_0, \ldots, x_n]_1$ and $(\lambda_1^j, \ldots, \lambda_k^j) \in \mathbb{F}^k - \{\underline{0}\}$ s.t.

$$f_j = \lambda_1^j \ell_1^{d_j} + \ldots + \lambda_k^j \ell_k^{d_j} \quad (2)$$

for all $j \in \{1, \ldots, r\}$, or in vector notation

$$f = \sum_{i=1}^{k} \left(\lambda_i^1 \ell_i^{d_1}, \ldots, \lambda_i^r \ell_i^{d_r} \right). \quad (3)$$

Let $n, r \in \mathbb{N}$ and let $d_1, \ldots, d_r \in \mathbb{N}$ such that $d_1 \leq \ldots \leq d_r$.
A *polynomial vector* $f = (f_1, \ldots, f_r)$ is a vector of homogeneous polynomials $f_j \in \mathbb{F}[x_0, \ldots, x_n]_{d_j}$, $j \in \{1, \ldots, r\}$.

Definition

A *(simultaneous) Waring decomposition* of $f = (f_1, \ldots, f_r)$ is given by $\ell_1, \ldots, \ell_k \in \mathbb{F}[x_0, \ldots, x_n]_1$ and $(\lambda_1^j, \ldots, \lambda_k^j) \in \mathbb{F}^k - \{\underline{0}\}$ s.t.

$$f_j = \lambda_1^j \ell_1^{d_j} + \ldots + \lambda_k^j \ell_k^{d_j} \tag{2}$$

for all $j \in \{1, \ldots, r\}$, or in vector notation

$$f = \sum_{i=1}^{k} \left(\lambda_i^1 \ell_i^{d_1}, \ldots, \lambda_i^r \ell_i^{d_r} \right). \tag{3}$$

Remark

In analogy to the case $r = 1$:

- the *(Waring) rank* of $f = (f_1, \dots, f_r)$ over $\mathbb{F}$ is the minimal k appearing in (3).
- the vector $f = (f_1, \dots, f_r)$ of general forms $f_j \in \mathbb{F}[x_0, \dots, x_n]_{d_j}$ is *Waring identifiable* over $\mathbb{F}$ if the presentation (3) is unique.

Theorem

A general $f = (f_1, \ldots, f_r)$ is Waring identifiable over $\mathbb{C}$ with rank k if it belongs to one of the following cases:

- $(n, r, d_1, \ldots, d_r, k) = \left(1, r, d_1, \ldots, d_r, \left\lceil \sum_{j=1}^{r} \binom{1+d_j}{d_j} \frac{1}{1+r} \right\rceil \right)$, with $d_1 + 1 \geq k$, [CR];
- $(n, r, d_1, \ldots, d_r, k) = (n, 2, 2, 2, n+1)$, [We];
- $(n, r, d_1, \ldots, d_r, k) = (2, 2, 2, 3, 4)$, [Ro];
- $(n, r, d_1, \ldots, d_r, k) = (2, 3, 3, 3, 4, 7)$, [AGMO];
- $(n, r, d_1, \ldots, d_r, k) = (2, 4, 2, 2, 2, 2, 4)$, Veronese.

Theorem

A general $f = (f_1, \dots, f_r)$ is Waring identifiable over $\mathbb{C}$ with rank k if it belongs to one of the following cases:

- $(n, r, d_1, \dots, d_r, k) = \left(1, r, d_1, \dots, d_r, \left\lceil \sum_{j=1}^{r} \binom{1+d_j}{d_j} \frac{1}{1+r} \right\rceil \right)$, with $d_1 + 1 \geq k$, [CR];
- $(n, r, d_1, \dots, d_r, k) = (n, 2, 2, 2, n+1)$, [We];
- $(n, r, d_1, \dots, d_r, k) = (2, 2, 2, 3, 4)$, [Ro];
- $(n, r, d_1, \dots, d_r, k) = (2, 3, 3, 3, 4, 7)$, [AGMO];
- $(n, r, d_1, \dots, d_r, k) = (2, 4, 2, 2, 2, 2, 4)$, Veronese.

Theorem

A general $f = (f_1, \ldots, f_r)$ is Waring identifiable over $\mathbb{C}$ with rank k if it belongs to one of the following cases:

- $(n, r, d_1, \ldots, d_r, k) = \left(1, r, d_1, \ldots, d_r, \left\lceil \sum_{j=1}^{r} \binom{1+d_j}{d_j} \frac{1}{1+r} \right\rceil\right)$, with $d_1 + 1 \geq k$, [CR];
- $(n, r, d_1, \ldots, d_r, k) = (n, 2, 2, 2, n+1)$, [We];
- $(n, r, d_1, \ldots, d_r, k) = (2, 2, 2, 3, 4)$, [Ro];
- $(n, r, d_1, \ldots, d_r, k) = (2, 3, 3, 3, 4, 7)$, [AGMO];
- $(n, r, d_1, \ldots, d_r, k) = (2, 4, 2, 2, 2, 2, 4)$, Veronese.

Theorem

A general $f = (f_1, \ldots, f_r)$ is Waring identifiable over $\mathbb{C}$ with rank k if it belongs to one of the following cases:

- $(n, r, d_1, \ldots, d_r, k) = \left(1, r, d_1, \ldots, d_r, \left\lceil \sum_{j=1}^{r} \binom{1+d_j}{d_j} \frac{1}{1+r} \right\rceil\right)$, with $d_1 + 1 \geq k$, [CR];
- $(n, r, d_1, \ldots, d_r, k) = (n, 2, 2, 2, n+1)$, [We];
- $(n, r, d_1, \ldots, d_r, k) = (2, 2, 2, 3, 4)$, [Ro];
- $(n, r, d_1, \ldots, d_r, k) = (2, 3, 3, 3, 4, 7)$, [AGMO];
- $(n, r, d_1, \ldots, d_r, k) = (2, 4, 2, 2, 2, 2, 4)$, Veronese.

Theorem

A general $f = (f_1, \ldots, f_r)$ is Waring identifiable over $\mathbb{C}$ with rank k if it belongs to one of the following cases:

- $(n, r, d_1, \ldots, d_r, k) = \left(1, r, d_1, \ldots, d_r, \left\lceil \sum_{j=1}^{r} \binom{1+d_j}{d_j} \frac{1}{1+r} \right\rceil \right)$, with $d_1 + 1 \geq k$, [CR];
- $(n, r, d_1, \ldots, d_r, k) = (n, 2, 2, 2, n+1)$, [We];
- $(n, r, d_1, \ldots, d_r, k) = (2, 2, 2, 3, 4)$, [Ro];
- $(n, r, d_1, \ldots, d_r, k) = (2, 3, 3, 3, 4, 7)$, [AGMO];
- $(n, r, d_1, \ldots, d_r, k) = (2, 4, 2, 2, 2, 2, 4)$, Veronese.

Theorem

A general $f = (f_1, \ldots, f_r)$ is Waring identifiable over $\mathbb{C}$ with rank k if it belongs to one of the following cases:

- $(n, r, d_1, \ldots, d_r, k) = \left(1, r, d_1, \ldots, d_r, \left\lceil \sum_{j=1}^{r} \binom{1+d_j}{d_j} \frac{1}{1+r} \right\rceil\right)$, with $d_1 + 1 \geq k$, [CR];
- $(n, r, d_1, \ldots, d_r, k) = (n, 2, 2, 2, n+1)$, [We];
- $(n, r, d_1, \ldots, d_r, k) = (2, 2, 2, 3, 4)$, [Ro];
- $(n, r, d_1, \ldots, d_r, k) = (2, 3, 3, 3, 4, 7)$, [AGMO];
- $(n, r, d_1, \ldots, d_r, k) = (2, 4, 2, 2, 2, 2, 4)$, Veronese.

Consider a general $p \in \mathbb{R}[x_0, \ldots, x_n]_d$.

Question

If p is not Waring identifiable over $\mathbb{C}$, what can one say about the Waring identifiability of p over $\mathbb{R}$?

Examples from [ABC]

Real identifiability holds in non trivial euclidean open subsets of $\mathbb{R}[x_0, \ldots, x_n]_d$ with:

- (n,d,k) = (2,7,12) ($\sharp\{$dec. over $\mathbb{C}\} = 5$, [DS]);
- (n,d,k) = (2,8,15) ($\sharp\{$dec. over $\mathbb{C}\} = 16$, [RS]).

Recent studies on the topic: [CLQ], [COV2], [DDL], [MMSV].

Consider a general $p \in \mathbb{R}[x_0, \dots, x_n]_d$.

Question

If p is not Waring identifiable over $\mathbb{C}$, what can one say about the Waring identifiability of p over $\mathbb{R}$?

Examples from [ABC]

Real identifiability holds in non trivial euclidean open subsets of $\mathbb{R}[x_0, \dots, x_n]_d$ with:

- $(n,d,k) = (2,7,12)$ ($\sharp\{$dec. over $\mathbb{C}\} = 5$, [DS]);
- $(n,d,k) = (2,8,15)$ ($\sharp\{$dec. over $\mathbb{C}\} = 16$, [RS]).

Recent studies on the topic: [CLQ], [COV2], [DDL], [MMSV].

Consider a general $p \in \mathbb{R}[x_0, \ldots, x_n]_d$.

Question

If p is not Waring identifiable over $\mathbb{C}$, what can one say about the Waring identifiability of p over $\mathbb{R}$?

Examples from [ABC]

Real identifiability holds in non trivial euclidean open subsets of $\mathbb{R}[x_0, \ldots, x_n]_d$ with:

- (n,d,k) $=$ (2,7,12) ($\sharp\{$dec. over $\mathbb{C}\} = 5$, [DS]);
- (n,d,k) $=$ (2,8,15) ($\sharp\{$dec. over $\mathbb{C}\} = 16$, [RS]).

Recent studies on the topic: [CLQ], [COV2], [DDL], [MMSV].

Theorem (A. 2017, [A])

Let $\mathbb{P}^{35}_{\mathbb{R}}$ be the projective space over $\mathbb{R}$ defined by real polynomial vectors with $n = 2$, $r = 4$ and $d_1 = 2, d_2 = d_3 = d_4 = 3$.
There exists a nontrivial Euclidean open set $U \subset \mathbb{P}^{35}_{\mathbb{R}}$ such that any $f \in U$ has rank 6 over $\mathbb{C}$ and is identifiable over $\mathbb{R}$ but not over $\mathbb{C}$.

Remark

The rank over $\mathbb{R}$ of any $f \in U$ is 6.

This result arises from a computational approach via the software Bertini ([Be], [BHSW]) for *Numerical Algebraic Geometry*.

Theorem (A. 2017, [A])

Let $\mathbb{P}^{35}_{\mathbb{R}}$ be the projective space over $\mathbb{R}$ defined by real polynomial vectors with $n = 2$, $r = 4$ and $d_1 = 2, d_2 = d_3 = d_4 = 3$.
There exists a nontrivial Euclidean open set $U \subset \mathbb{P}^{35}_{\mathbb{R}}$ such that any $f \in U$ has rank 6 over $\mathbb{C}$ and is identifiable over $\mathbb{R}$ but not over $\mathbb{C}$.

Remark

The rank over $\mathbb{R}$ of any $f \in U$ is 6.

This result arises from a computational approach via the software Bertini ([Be], [BHSW]) for *Numerical Algebraic Geometry*.

Theorem (A. 2017, [A])

Let $\mathbb{P}^{35}_{\mathbb{R}}$ be the projective space over $\mathbb{R}$ defined by real polynomial vectors with $n = 2$, $r = 4$ and $d_1 = 2, d_2 = d_3 = d_4 = 3$.
There exists a nontrivial Euclidean open set $U \subset \mathbb{P}^{35}_{\mathbb{R}}$ such that any $f \in U$ has rank 6 over $\mathbb{C}$ and is identifiable over $\mathbb{R}$ but not over $\mathbb{C}$.

Remark

The rank over $\mathbb{R}$ of any $f \in U$ is 6.

This result arises from a computational approach via the software Bertini ([Be], [BHSW]) for *Numerical Algebraic Geometry*.

Consider the polynomial system

$$\begin{cases} f_1 - \lambda_1^1 \ell_1^2 - \ldots - \lambda_6^1 \ell_6^2 = 0 \\ f_2 - \lambda_1^2 \ell_1^3 - \ldots - \lambda_6^2 \ell_6^3 = 0 \\ f_3 - \lambda_1^3 \ell_1^3 - \ldots - \lambda_6^3 \ell_6^3 = 0 \\ f_4 - \lambda_1^4 \ell_1^3 - \ldots - \lambda_6^4 \ell_6^3 = 0 \end{cases} \tag{4}$$

where:

- $f_j \in \mathbb{R}[x_0, x_1, x_2]_{d_j}$ known, $j \in \{1, \ldots, 4\}$;
- $\ell_i = x_0 + l_1^i x_1 + l_2^i x_2 \in \mathbb{C}[x_0, x_1, x_2]_1$ unknown, $i \in \{1, \ldots, 6\}$;
- $\lambda_i^j \in \mathbb{C}$ unknown, $i \in \{1, \ldots, 6\}$, $j \in \{1, \ldots, 4\}$.

$F_{(f_1, f_2, f_3, f_4)}\left(\left[l_1^1, l_2^1, \lambda_1^1, \lambda_1^2, \lambda_1^3, \lambda_1^4\right], \ldots, \left[l_1^6, l_2^6, \lambda_6^1, \lambda_6^2, \lambda_6^3, \lambda_6^4\right]\right)$: square nonlinear system of order 36 arising from (4).

Consider the polynomial system

$$\left\{\begin{array}{l} f_1 - \lambda_1^1 \ell_1^2 - \ldots - \lambda_6^1 \ell_6^2 = 0 \\ f_2 - \lambda_1^2 \ell_1^3 - \ldots - \lambda_6^2 \ell_6^3 = 0 \\ f_3 - \lambda_1^3 \ell_1^3 - \ldots - \lambda_6^3 \ell_6^3 = 0 \\ f_4 - \lambda_1^4 \ell_1^3 - \ldots - \lambda_6^4 \ell_6^3 = 0 \end{array}\right. \tag{4}$$

where:

- $f_j \in \mathbb{R}[x_0, x_1, x_2]_{d_j}$ known, $j \in \{1, \ldots, 4\}$;
- $\ell_i = x_0 + l_1^i x_1 + l_2^i x_2 \in \mathbb{C}[x_0, x_1, x_2]_1$ unknown, $i \in \{1, \ldots, 6\}$;
- $\lambda_i^j \in \mathbb{C}$ unknown, $i \in \{1, \ldots, 6\}$, $j \in \{1, \ldots, 4\}$.

$F_{(f_1,f_2,f_3,f_4)}\left(\left[l_1^1, l_2^1, \lambda_1^1, \lambda_1^2, \lambda_1^3, \lambda_1^4\right], \ldots, \left[l_1^6, l_2^6, \lambda_6^1, \lambda_6^2, \lambda_6^3, \lambda_6^4\right]\right)$: square nonlinear system of order 36 arising from (4).

Consider the polynomial system

$$\begin{cases} f_1 - \lambda_1^1 \ell_1^2 - \ldots - \lambda_6^1 \ell_6^2 = 0 \\ f_2 - \lambda_1^2 \ell_1^3 - \ldots - \lambda_6^2 \ell_6^3 = 0 \\ f_3 - \lambda_1^3 \ell_1^3 - \ldots - \lambda_6^3 \ell_6^3 = 0 \\ f_4 - \lambda_1^4 \ell_1^3 - \ldots - \lambda_6^4 \ell_6^3 = 0 \end{cases} \qquad (4)$$

where:

- $f_j \in \mathbb{R}[x_0, x_1, x_2]_{d_j}$ known, $j \in \{1, \ldots, 4\}$;
- $\ell_i = x_0 + l_1^i x_1 + l_2^i x_2 \in \mathbb{C}[x_0, x_1, x_2]_1$ unknown, $i \in \{1, \ldots, 6\}$;
- $\lambda_i^j \in \mathbb{C}$ unknown, $i \in \{1, \ldots, 6\}$, $j \in \{1, \ldots, 4\}$.

$F_{(f_1,f_2,f_3,f_4)}\left(\left[l_1^1, l_2^1, \lambda_1^1, \lambda_1^2, \lambda_1^3, \lambda_1^4\right], \ldots, \left[l_1^6, l_2^6, \lambda_6^1, \lambda_6^2, \lambda_6^3, \lambda_6^4\right]\right)$: square nonlinear system of order 36 arising from (4).

AIM: compute the number of real solutions of $F_{(f_1,f_2,f_3,f_4)}$

INPUT

- $\left(\left[\bar{l}_1^1, \bar{l}_2^1, \bar{\lambda}_1^1, \bar{\lambda}_1^2, \bar{\lambda}_1^3, \bar{\lambda}_1^4\right], \ldots, \left[\bar{l}_1^6, \bar{l}_2^6, \bar{\lambda}_6^1, \bar{\lambda}_6^2, \bar{\lambda}_6^3, \bar{\lambda}_6^4\right]\right) \in \mathbb{R}^{36}$ startpoint
- $\bar{f} = (\bar{f}_1, \bar{f}_2, \bar{f}_3, \bar{f}_4) \in \mathbb{R}^{36}$ startparameters

TRIANGLE-LOOP

1) startpar. $\Rightarrow$ finalparameters $=$ random element of $\mathbb{C}^{36} \Rightarrow F_1$
$H_0 : \mathbb{C}^{36} \times [0,1] \to \mathbb{C}^{36}$ segment homotopy $F_{(\bar{f}_1,\bar{f}_2,\bar{f}_3,\bar{f}_4)} \rightsquigarrow F_1$
startpoint $=$ sol. of $F_{(\bar{f}_1,\bar{f}_2,\bar{f}_3,\bar{f}_4)} \rightsquigarrow$ endpoint $=$ sol. of F_1

AIM: compute the number of real solutions of $F_{(f_1,f_2,f_3,f_4)}$

INPUT

- $\left(\left[\overline{l}_1^1, \overline{l}_2^1, \overline{\lambda}_1^1, \overline{\lambda}_1^2, \overline{\lambda}_1^3, \overline{\lambda}_1^4\right], \ldots, \left[\overline{l}_1^6, \overline{l}_2^6, \overline{\lambda}_6^1, \overline{\lambda}_6^2, \overline{\lambda}_6^3, \overline{\lambda}_6^4\right]\right) \in \mathbb{R}^{36}$ startpoint
- $\overline{f} = (\overline{f}_1, \overline{f}_2, \overline{f}_3, \overline{f}_4) \in \mathbb{R}^{36}$ startparameters

TRIANGLE-LOOP

1) startpar. $\Rightarrow$ finalparameters $=$ random element of $\mathbb{C}^{36} \Rightarrow F_1$
$H_0 : \mathbb{C}^{36} \times [0,1] \to \mathbb{C}^{36}$ segment homotopy $F_{(\overline{f}_1,\overline{f}_2,\overline{f}_3,\overline{f}_4)} \rightsquigarrow F_1$
startpoint $=$ sol. of $F_{(\overline{f}_1,\overline{f}_2,\overline{f}_3,\overline{f}_4)} \rightsquigarrow$ endpoint $=$ sol. of F_1

AIM: compute the number of real solutions of $F_{(f_1,f_2,f_3,f_4)}$

INPUT

- $\left(\left[\overline{l}^1_1, \overline{l}^1_2, \overline{\lambda}^1_1, \overline{\lambda}^2_1, \overline{\lambda}^3_1, \overline{\lambda}^4_1\right], \dots, \left[\overline{l}^6_1, \overline{l}^6_2, \overline{\lambda}^1_6, \overline{\lambda}^2_6, \overline{\lambda}^3_6, \overline{\lambda}^4_6\right]\right) \in \mathbb{R}^{36}$ startpoint
- $\overline{f} = (\overline{f}_1, \overline{f}_2, \overline{f}_3, \overline{f}_4) \in \mathbb{R}^{36}$ startparameters

TRIANGLE-LOOP

1) startpar. $\Rightarrow$ finalparameters $=$ random element of $\mathbb{C}^{36} \Rightarrow F_1$

$H_0 : \mathbb{C}^{36} \times [0,1] \to \mathbb{C}^{36}$ segment homotopy $F_{(\overline{f}_1,\overline{f}_2,\overline{f}_3,\overline{f}_4)} \rightsquigarrow F_1$

startpoint $=$ sol. of $F_{(\overline{f}_1,\overline{f}_2,\overline{f}_3,\overline{f}_4)} \rightsquigarrow$ endpoint $=$ sol. of F_1

AIM: compute the number of real solutions of $F_{(f_1,f_2,f_3,f_4)}$

INPUT

- $\left(\left[\overline{l}_1^1,\overline{l}_2^1,\overline{\lambda}_1^1,\overline{\lambda}_1^2,\overline{\lambda}_1^3,\overline{\lambda}_1^4\right],\ldots,\left[\overline{l}_1^6,\overline{l}_2^6,\overline{\lambda}_6^1,\overline{\lambda}_6^2,\overline{\lambda}_6^3,\overline{\lambda}_6^4\right]\right)\in\mathbb{R}^{36}$ startpoint
- $\overline{f}=(\overline{f}_1,\overline{f}_2,\overline{f}_3,\overline{f}_4)\in\mathbb{R}^{36}$ startparameters

TRIANGLE-LOOP

1) startpar. $\Rightarrow$ finalparameters $=$ random element of $\mathbb{C}^{36}\Rightarrow F_1$

$H_0:\mathbb{C}^{36}\times[0,1]\to\mathbb{C}^{36}$ segment homotopy $F_{(\overline{f}_1,\overline{f}_2,\overline{f}_3,\overline{f}_4)}\rightsquigarrow F_1$

startpoint $=$ sol. of $F_{(\overline{f}_1,\overline{f}_2,\overline{f}_3,\overline{f}_4)}\rightsquigarrow$ endpoint $=$ sol. of F_1

2) startpar. $\Rightarrow$ finalpar. = random element of $\mathbb{C}^{36} \Rightarrow F_2$

$H_1 : \mathbb{C}^{36} \times [0,1] \to \mathbb{C}^{36}$ segment homotopy $F_1 \rightsquigarrow F_2$

startpoint = sol. of $F_1 \rightsquigarrow$ endpoint = sol. of F_2

3) startpar. $\Rightarrow \overline{f} = (\overline{f}_1, \overline{f}_2, \overline{f}_3, \overline{f}_4) \Rightarrow F_{(\overline{f}_1, \overline{f}_2, \overline{f}_3, \overline{f}_4)}$

$H_2 : \mathbb{C}^{36} \times [0,1] \to \mathbb{C}^{36}$ segment homotopy $F_2 \rightsquigarrow F_{(\overline{f}_1, \overline{f}_2, \overline{f}_3, \overline{f}_4)}$

startpoint = sol. of $F_2 \rightsquigarrow$ endpoint = sol. of $F_{(\overline{f}_1, \overline{f}_2, \overline{f}_3, \overline{f}_4)}$

OUTPUT

- A Waring decomposition of $\overline{f} = (\overline{f}_1, \overline{f}_2, \overline{f}_3, \overline{f}_4)$

2) startpar. $\Rightarrow$ finalpar. $=$ random element of $\mathbb{C}^{36} \Rightarrow F_2$

$H_1 : \mathbb{C}^{36} \times [0,1] \to \mathbb{C}^{36}$ segment homotopy $F_1 \rightsquigarrow F_2$

startpoint $=$ sol. of $F_1 \rightsquigarrow$ endpoint $=$ sol. of F_2

3) startpar. $\Rightarrow \overline{f} = (\overline{f}_1, \overline{f}_2, \overline{f}_3, \overline{f}_4) \Rightarrow F_{(\overline{f}_1,\overline{f}_2,\overline{f}_3,\overline{f}_4)}$

$H_2 : \mathbb{C}^{36} \times [0,1] \to \mathbb{C}^{36}$ segment homotopy $F_2 \rightsquigarrow F_{(\overline{f}_1,\overline{f}_2,\overline{f}_3,\overline{f}_4)}$

startpoint $=$ sol. of $F_2 \rightsquigarrow$ endpoint $=$ sol. of $F_{(\overline{f}_1,\overline{f}_2,\overline{f}_3,\overline{f}_4)}$

OUTPUT

- A Waring decomposition of $\overline{f} = (\overline{f}_1, \overline{f}_2, \overline{f}_3, \overline{f}_4)$

2) startpar. $\Rightarrow$ finalpar. $=$ random element of $\mathbb{C}^{36} \Rightarrow F_2$

$H_1 : \mathbb{C}^{36} \times [0,1] \to \mathbb{C}^{36}$ segment homotopy $F_1 \rightsquigarrow F_2$

startpoint $=$ sol. of $F_1 \rightsquigarrow$ endpoint $=$ sol. of F_2

3) startpar. $\Rightarrow \overline{f} = (\overline{f}_1, \overline{f}_2, \overline{f}_3, \overline{f}_4) \Rightarrow F_{(\overline{f}_1,\overline{f}_2,\overline{f}_3,\overline{f}_4)}$

$H_2 : \mathbb{C}^{36} \times [0,1] \to \mathbb{C}^{36}$ segment homotopy $F_2 \rightsquigarrow F_{(\overline{f}_1,\overline{f}_2,\overline{f}_3,\overline{f}_4)}$

startpoint $=$ sol. of $F_2 \rightsquigarrow$ endpoint $=$ sol. of $F_{(\overline{f}_1,\overline{f}_2,\overline{f}_3,\overline{f}_4)}$

OUTPUT

- A Waring decomposition of $\overline{f} = (\overline{f}_1, \overline{f}_2, \overline{f}_3, \overline{f}_4)$

CONCLUSIONS

- output $= \left(\left[\overline{l}_1^1, \overline{l}_2^1, \overline{\lambda}_1^1, \overline{\lambda}_1^2, \overline{\lambda}_1^3, \overline{\lambda}_1^4\right], \ldots, \left[\overline{l}_1^6, \overline{l}_2^6, \overline{\lambda}_6^1, \overline{\lambda}_6^2, \overline{\lambda}_6^3, \overline{\lambda}_6^4\right]\right)$
  $\Rightarrow$ restart the triangle-loop
- otherwise:
  - output $\in \mathbb{R}^{36} \Rightarrow \overline{f} = (\overline{f}_1, \overline{f}_2, \overline{f}_3, \overline{f}_4)$ is not real identifiable
  - output $\in \mathbb{C}^{36} - \mathbb{R}^{36} \Rightarrow \overline{f} = (\overline{f}_1, \overline{f}_2, \overline{f}_3, \overline{f}_4)$ is real identifiable

Remark

This procedure provides an answer, since the general complex polynomial vector $f = (f_1, f_2, f_3, f_4)$ has 2 Waring decompositions with rank 6.

CONCLUSIONS

- output $= \left(\left[\bar{l}_1^1, \bar{l}_2^1, \bar{\lambda}_1^1, \bar{\lambda}_1^2, \bar{\lambda}_1^3, \bar{\lambda}_1^4 \right], \dots, \left[\bar{l}_1^6, \bar{l}_2^6, \bar{\lambda}_6^1, \bar{\lambda}_6^2, \bar{\lambda}_6^3, \bar{\lambda}_6^4 \right] \right)$
  $\Rightarrow$ restart the triangle-loop
- otherwise:
  - ▷ output $\in \mathbb{R}^{36} \Rightarrow \bar{f} = (\bar{f}_1, \bar{f}_2, \bar{f}_3, \bar{f}_4)$ is not real identifiable
  - ▷ output $\in \mathbb{C}^{36} - \mathbb{R}^{36} \Rightarrow \bar{f} = (\bar{f}_1, \bar{f}_2, \bar{f}_3, \bar{f}_4)$ is real identifiable

Remark

This procedure provides an answer, since the general complex polynomial vector $f = (f_1, f_2, f_3, f_4)$ has 2 Waring decompositions with rank 6.

CONCLUSIONS

- output $= \left(\left[\overline{l}_1^1, \overline{l}_2^1, \overline{\lambda}_1^1, \overline{\lambda}_1^2, \overline{\lambda}_1^3, \overline{\lambda}_1^4\right], \dots, \left[\overline{l}_1^6, \overline{l}_2^6, \overline{\lambda}_6^1, \overline{\lambda}_6^2, \overline{\lambda}_6^3, \overline{\lambda}_6^4\right]\right)$
  $\Rightarrow$ restart the triangle-loop
- otherwise:
  - ▷ output $\in \mathbb{R}^{36} \Rightarrow \overline{f} = (\overline{f}_1, \overline{f}_2, \overline{f}_3, \overline{f}_4)$ is not real identifiable
  - ▷ output $\in \mathbb{C}^{36} - \mathbb{R}^{36} \Rightarrow \overline{f} = (\overline{f}_1, \overline{f}_2, \overline{f}_3, \overline{f}_4)$ is real identifiable

Remark

This procedure provides an answer, since the general complex polynomial vector $f = (f_1, f_2, f_3, f_4)$ has 2 Waring decompositions with rank 6.

Apply the procedure with startpoint:

$$\begin{bmatrix} \bar{l}_1^1 \\ \bar{l}_2^1 \\ \bar{\lambda}_1^1 \\ \bar{\lambda}_1^2 \\ \bar{\lambda}_1^3 \\ \bar{\lambda}_1^4 \end{bmatrix} = \begin{bmatrix} -8.917867308121220 \\ -3.882527532557760 \\ 3.802256643629860 \\ 8.752472902826010 \\ -5.012502093548760 \\ 4.499965370038940 \end{bmatrix} \cdot 10^{-1}$$

$$\begin{bmatrix} \bar{l}_1^2 \\ \bar{l}_2^2 \\ \bar{\lambda}_2^1 \\ \bar{\lambda}_2^2 \\ \bar{\lambda}_2^3 \\ \bar{\lambda}_2^4 \end{bmatrix} = \begin{bmatrix} 7.119541680224431 \\ -4.656057771083370 \\ -5.411549110491870 \\ -6.257070109519970 \\ 2.294191400826210 \\ -3.750700782420560 \end{bmatrix} \cdot 10^{-1}$$

$$\begin{bmatrix} \bar{l}_1^3 \\ \bar{l}_2^3 \\ \bar{\lambda}_3^1 \\ \bar{\lambda}_3^2 \\ \bar{\lambda}_3^3 \\ \bar{\lambda}_3^4 \end{bmatrix} = \begin{bmatrix} 8.830647057133760 \\ -5.026318665541210 \\ -9.252016941613800 \\ 8.662890126034720 \\ -7.498635837727520 \\ -7.436378620768880 \end{bmatrix} \cdot 10^{-1}; \quad \begin{bmatrix} \bar{l}_1^4 \\ \bar{l}_2^4 \\ \bar{\lambda}_4^1 \\ \bar{\lambda}_4^2 \\ \bar{\lambda}_4^3 \\ \bar{\lambda}_4^4 \end{bmatrix} = \begin{bmatrix} -5.037776998327460 \\ 7.225195917485460 \\ 7.366967625095110 \\ 9.363491701837950 \\ -3.075057135421950 \\ -4.554394894892720 \end{bmatrix} \cdot 10^{-1}$$

$$\begin{bmatrix} \bar{l}_5^2 \\ \bar{l}_5^2 \\ \bar{\lambda}_5^1 \\ \bar{\lambda}_5^2 \\ \bar{\lambda}_5^3 \\ \bar{\lambda}_5^4 \end{bmatrix} = \begin{bmatrix} 3.901539106331100 \\ 3.211377682018530 \\ 1.637407925571110 \\ 7.404821190992460 \\ -7.815152102127080 \\ 6.417745789016570 \end{bmatrix} \cdot 10^{-1}; \quad \begin{bmatrix} \bar{l}_6^3 \\ \bar{l}_6^3 \\ \bar{\lambda}_6^1 \\ \bar{\lambda}_6^2 \\ \bar{\lambda}_6^3 \\ \bar{\lambda}_6^4 \end{bmatrix} = \begin{bmatrix} -5.295293887908490 \\ -5.561058003667060 \\ -9.944247658494601 \\ -5.708459618609850 \\ 6.892129936226110 \\ 9.841403826077800 \end{bmatrix} \cdot 10^{-1}$$

The solutions of $F_{(\overline{f}_1,\overline{f}_2,\overline{f}_3,\overline{f}_4)}$ are the startpoint and a non-real *self-conjugate* point:

$$\begin{bmatrix} l_1^1 \\ l_2^1 \\ \lambda_1^1 \\ \lambda_1^2 \\ \lambda_1^3 \\ \lambda_1^4 \end{bmatrix} = \begin{bmatrix} -6.686660607487932 \cdot 10^{-1} + i\,2.991560236304532 \cdot 10^{-1} \\ -4.002544019334531 \cdot 10^{-1} + i\,5.280306195219122 \cdot 10^{-2} \\ 1.594553827837448 \cdot 10^{-1} + i\,3.356599720053134 \cdot 10^{-1} \\ 4.446250113129167 \cdot 10^{-1} + i\,6.673927781556271 \cdot 10^{-1} \\ -2.454862075682181 \cdot 10^{-1} - i\,3.983503531385789 \cdot 10^{-1} \\ 2.764690785411610 \cdot 10^{-1} + i\,2.158521779288920 \cdot 10^{-1} \end{bmatrix}$$

$$\begin{bmatrix} l_1^2 \\ l_2^2 \\ \lambda_2^1 \\ \lambda_2^2 \\ \lambda_2^3 \\ \lambda_2^4 \end{bmatrix} = \begin{bmatrix} -6.686660607487904 \cdot 10^{-1} - i\,2.991560236304547 \cdot 10^{-1} \\ -4.002544019334515 \cdot 10^{-1} - i\,5.280306195219232 \cdot 10^{-2} \\ 1.594553827837373 \cdot 10^{-1} - i\,3.356599720053096 \cdot 10^{-1} \\ 4.446250113129012 \cdot 10^{-1} - i\,6.673927781556222 \cdot 10^{-1} \\ -2.454862075682089 \cdot 10^{-1} + i\,3.983503531385751 \cdot 10^{-1} \\ 2.764690785411556 \cdot 10^{-1} - i\,2.158521779288923 \cdot 10^{-1} \end{bmatrix}$$

$$\begin{bmatrix} l_1^3 \\ l_2^3 \\ \lambda_3^1 \\ \lambda_3^2 \\ \lambda_3^3 \\ \lambda_3^4 \end{bmatrix} = \begin{bmatrix} 8.570579460266361 \cdot 10^{-1} - i\,7.283128093671376 \cdot 10^{-17} \\ -5.008030666483697 \cdot 10^{-1} - i\,4.008159906407349 \cdot 10^{-17} \\ -1.367408916610990 + i\,3.306816626080789 \cdot 10^{-18} \\ 4.536429737440653 \cdot 10^{-1} + i\,2.910405206765776 \cdot 10^{-17} \\ -6.192887442009833 \cdot 10^{-1} + i\,4.725766219321192 \cdot 10^{-17} \\ -1.039617887248617 - i\,2.495155774703828 \cdot 10^{-16} \end{bmatrix}$$

$$\begin{bmatrix} l_1^4 \\ l_2^4 \\ \lambda_4^1 \\ \lambda_4^2 \\ \lambda_4^3 \\ \lambda_4^4 \end{bmatrix} = \begin{bmatrix} -5.699778692822882 \cdot 10^{-1} + i\,4.865357249028701 \cdot 10^{-17} \\ -5.677137601850224 \cdot 10^{-1} + i\,2.960617319879011 \cdot 10^{-16} \\ -1.044756004401222 - i\,3.887420441973932 \cdot 10^{-15} \\ -8.377080722561171 \cdot 10^{-1} - i\,5.142777479913430 \cdot 10^{-15} \\ 8.135414334604811 \cdot 10^{-1} + i\,3.676434699208253 \cdot 10^{-15} \\ 7.921737192603013 \cdot 10^{-1} + i\,4.146260158127690 \cdot 10^{-16} \end{bmatrix}$$

$$\begin{bmatrix} l_1^5 \\ l_2^5 \\ \lambda_5^1 \\ \lambda_5^2 \\ \lambda_5^3 \\ \lambda_5^4 \end{bmatrix} = \begin{bmatrix} 3.708159733617827 \cdot 10^{-1} - i\, 7.774407203078870 \cdot 10^{-17} \\ 3.120387814866833 \cdot 10^{-1} - i\, 1.722526201536345 \cdot 10^{-17} \\ 1.284971228628584 \cdot 10^{-1} - i\, 1.377055343669206 \cdot 10^{-16} \\ 7.208233002630405 \cdot 10^{-1} - i\, 2.402185438413196 \cdot 10^{-16} \\ -7.991170011183902 \cdot 10^{-1} + i\, 3.506038775275000 \cdot 10^{-17} \\ 6.673476599865575 \cdot 10^{-1} + i\, 3.473241659557313 \cdot 10^{-16} \end{bmatrix}$$

$$\begin{bmatrix} l_1^6 \\ l_2^6 \\ \lambda_6^1 \\ \lambda_6^2 \\ \lambda_6^3 \\ \lambda_6^4 \end{bmatrix} = \begin{bmatrix} -4.832216683406109 \cdot 10^{-1} + i\, 1.233144445930701 \cdot 10^{-16} \\ 7.039108040428517 \cdot 10^{-1} - i\, 5.299038118022903 \cdot 10^{-18} \\ 7.846388809514525 \cdot 10^{-1} + i\, 2.899156609226239 \cdot 10^{-16} \\ 9.958063949793251 \cdot 10^{-1} + i\, 3.588980241470141 \cdot 10^{-16} \\ -3.256658561819794 \cdot 10^{-1} - i\, 9.299744134494414 \cdot 10^{-17} \\ -4.710775803754439 \cdot 10^{-1} - i\, 1.431079105045086 \cdot 10^{-16} \end{bmatrix}$$

Move $\overline{f} = (\overline{f}_1, \overline{f}_2, \overline{f}_3, \overline{f}_4)$ in a small Euclidean open subset of $\mathbb{P}^{35}_{\mathbb{R}}$
$\rightsquigarrow$ only one decomposition remains real $\rightsquigarrow$ get U

Theorem (A.-Bocci-Chiantini 2017, [ABC1])

A general $f = (f_1, f_2, f_3, f_4)$ s.t. $f_j \in \mathbb{C}[x_0, x_1, x_2]_3$ for $j \in \{2, 3, 4\}$ and $f_1 \in \mathbb{C}[x_0, x_1, x_2]_2$ has 2 Waring decompositions over $\mathbb{C}$ with rank 6.

Sketch of the proof

- Fix a decomposition of f given by $q_i = (\lambda_i^1 \ell_i^2, \lambda_i^2 \ell_i^3, \lambda_i^3 \ell_i^3, \lambda_i^4 \ell_i^3)$
- $C_f : (\mathrm{Sym}^2 \mathbb{C}^3)^\vee \to \mathbb{C} \oplus (\mathrm{Sym}^1 \mathbb{C}^3)^{\oplus 3}$ *catalecticant map* of the contraction by f
- C_f has rank 6
- $\mathbb{P}(\mathrm{im}(C_f)) = \langle [L_1], \ldots, [L_6] \rangle$, where $[L_i] \in C_V \,\mathrm{Seg}(\mathbb{P}^2 \times \mathbb{P}^2)$, $i \in \{1, \ldots, 6\}$, corresponds to q_i

Theorem (A.-Bocci-Chiantini 2017, [ABC1])

A general $f = (f_1, f_2, f_3, f_4)$ s.t. $f_j \in \mathbb{C}[x_0, x_1, x_2]_3$ for $j \in \{2, 3, 4\}$ and $f_1 \in \mathbb{C}[x_0, x_1, x_2]_2$ has 2 Waring decompositions over $\mathbb{C}$ with rank 6.

Sketch of the proof

- Fix a decomposition of f given by $q_i = (\lambda_i^1 \ell_i^2, \lambda_i^2 \ell_i^3, \lambda_i^3 \ell_i^3, \lambda_i^4 \ell_i^3)$
- $C_f : (\mathrm{Sym}^2\, \mathbb{C}^3)^\vee \to \mathbb{C} \oplus (\mathrm{Sym}^1\, \mathbb{C}^3)^{\oplus 3}$ *catalecticant map* of the contraction by f
- C_f has rank 6
- $\mathbb{P}(\mathrm{im}(C_f)) = \langle [L_1], \ldots, [L_6] \rangle$, where $[L_i] \in C_V\, \mathrm{Seg}(\mathbb{P}^2 \times \mathbb{P}^2)$, $i \in \{1, \ldots, 6\}$, corresponds to q_i

Theorem (A.-Bocci-Chiantini 2017, [ABC1])

A general $f = (f_1, f_2, f_3, f_4)$ s.t. $f_j \in \mathbb{C}[x_0, x_1, x_2]_3$ for $j \in \{2, 3, 4\}$ and $f_1 \in \mathbb{C}[x_0, x_1, x_2]_2$ has 2 Waring decompositions over $\mathbb{C}$ with rank 6.

Sketch of the proof

- Fix a decomposition of f given by $q_i = (\lambda_i^1 \ell_i^2, \lambda_i^2 \ell_i^3, \lambda_i^3 \ell_i^3, \lambda_i^4 \ell_i^3)$
- $C_f : (\mathrm{Sym}^2 \mathbb{C}^3)^\vee \to \mathbb{C} \oplus (\mathrm{Sym}^1 \mathbb{C}^3)^{\oplus 3}$ *catalecticant map* of the contraction by f
- C_f has rank 6
- $\mathbb{P}(\mathrm{im}(C_f)) = \langle [L_1], \ldots, [L_6] \rangle$, where $[L_i] \in C_V \,\mathrm{Seg}(\mathbb{P}^2 \times \mathbb{P}^2)$, $i \in \{1, \ldots, 6\}$, corresponds to q_i

Theorem (A.-Bocci-Chiantini 2017, [ABC1])

A general $f = (f_1, f_2, f_3, f_4)$ s.t. $f_j \in \mathbb{C}[x_0, x_1, x_2]_3$ for $j \in \{2, 3, 4\}$ and $f_1 \in \mathbb{C}[x_0, x_1, x_2]_2$ has 2 Waring decompositions over $\mathbb{C}$ with rank 6.

Sketch of the proof

- Fix a decomposition of f given by $q_i = (\lambda_i^1 \ell_i^2, \lambda_i^2 \ell_i^3, \lambda_i^3 \ell_i^3, \lambda_i^4 \ell_i^3)$
- $C_f : (\mathrm{Sym}^2\, \mathbb{C}^3)^\vee \to \mathbb{C} \oplus (\mathrm{Sym}^1\, \mathbb{C}^3)^{\oplus 3}$ *catalecticant map* of the contraction by f
- C_f has rank 6
- $\mathbb{P}(\mathrm{im}(C_f)) = \langle [L_1], \dots, [L_6] \rangle$, where $[L_i] \in C_V\, \mathrm{Seg}(\mathbb{P}^2 \times \mathbb{P}^2)$, $i \in \{1, \dots, 6\}$, corresponds to q_i

Theorem (A.-Bocci-Chiantini 2017, [ABC1])

A general $f = (f_1, f_2, f_3, f_4)$ s.t. $f_j \in \mathbb{C}[x_0, x_1, x_2]_3$ for $j \in \{2, 3, 4\}$ and $f_1 \in \mathbb{C}[x_0, x_1, x_2]_2$ has 2 Waring decompositions over $\mathbb{C}$ with rank 6.

Sketch of the proof

- Fix a decomposition of f given by $q_i = (\lambda_i^1 \ell_i^2, \lambda_i^2 \ell_i^3, \lambda_i^3 \ell_i^3, \lambda_i^4 \ell_i^3)$
- $C_f : (\mathrm{Sym}^2\, \mathbb{C}^3)^\vee \to \mathbb{C} \oplus (\mathrm{Sym}^1\, \mathbb{C}^3)^{\oplus 3}$ *catalecticant map* of the contraction by f
- C_f has rank 6
- $\mathbb{P}(\mathrm{im}(C_f)) = \langle [L_1], \ldots, [L_6] \rangle$, where $[L_i] \in C_V\, \mathrm{Seg}(\mathbb{P}^2 \times \mathbb{P}^2)$, $i \in \{1, \ldots, 6\}$, corresponds to q_i

Sketch of the proof

- $\mathcal{C}_6 = \mathbb{P}(\mathrm{im}(C_f)) \cap C_V \, \mathrm{Seg}(\mathbb{P}^2 \times \mathbb{P}^2)$ is an elliptic normal curve of degree 6 $\rightsquigarrow \mathcal{C}_{12}$ elliptic normal curve of degree 12
- $q_1, \ldots, q_6 \in \mathcal{C}_{12}$
- $\mathcal{C}_{12}$ has 2 6-secant spaces containing f

□

Remark

The Theorem can be extended to general polynomial vectors with 3 cubics and an arbitrary number of conics.

Sketch of the proof

- $\mathcal{C}_6 = \mathbb{P}(\mathrm{im}(C_f)) \cap C_V \, \mathrm{Seg}(\mathbb{P}^2 \times \mathbb{P}^2)$ is an elliptic normal curve of degree 6 $\rightsquigarrow \mathcal{C}_{12}$ elliptic normal curve of degree 12
- $q_1, \ldots, q_6 \in \mathcal{C}_{12}$
- $\mathcal{C}_{12}$ has 2 6-secant spaces containing f

□

Remark

The Theorem can be extended to general polynomial vectors with 3 cubics and an arbitrary number of conics.

Sketch of the proof

- $\mathcal{C}_6 = \mathbb{P}(\mathrm{im}(C_f)) \cap C_V\,\mathrm{Seg}(\mathbb{P}^2 \times \mathbb{P}^2)$ is an elliptic normal curve of degree 6 $\rightsquigarrow \mathcal{C}_{12}$ elliptic normal curve of degree 12
- $q_1, \ldots, q_6 \in \mathcal{C}_{12}$
- $\mathcal{C}_{12}$ has 2 6-secant spaces containing f

□

Remark

The Theorem can be extended to general polynomial vectors with 3 cubics and an arbitrary number of conics.

Sketch of the proof

- $\mathcal{C}_6 = \mathbb{P}(\mathrm{im}(C_f)) \cap C_V\,\mathrm{Seg}(\mathbb{P}^2 \times \mathbb{P}^2)$ is an elliptic normal curve of degree 6 $\rightsquigarrow \mathcal{C}_{12}$ elliptic normal curve of degree 12
- $q_1, \ldots, q_6 \in \mathcal{C}_{12}$
- $\mathcal{C}_{12}$ has 2 6-secant spaces containing f

$\square$

Remark

The Theorem can be extended to general polynomial vectors with 3 cubics and an arbitrary number of conics.

Sketch of the proof

- $\mathcal{C}_6 = \mathbb{P}(\mathrm{im}(C_f)) \cap C_V \,\mathrm{Seg}(\mathbb{P}^2 \times \mathbb{P}^2)$ is an elliptic normal curve of degree 6 $\rightsquigarrow \mathcal{C}_{12}$ elliptic normal curve of degree 12
- $q_1, \ldots, q_6 \in \mathcal{C}_{12}$
- $\mathcal{C}_{12}$ has 2 6-secant spaces containing f

□

Remark

The Theorem can be extended to general polynomial vectors with 3 cubics and an arbitrary number of conics.

A (simultaneous) Waring decomposition has *sub-generic rank k* if:

$$k < \frac{\sum_{j=1}^{r} \binom{n+d_j}{d_j}}{n+r}. \tag{5}$$

Hessian criterion based on a generalization of Lemma 5.1, [COV1].

Lemma (Sufficient condition for identifiability)

Let $X^n_{d_1,\ldots,d_r} = \mathbb{P}(\bigoplus_{j=1}^r \mathcal{O}_{\mathbb{P}^n_{\mathbb{C}}}(d_j))$ be not k-defective, with k as in (5). Let $g_1, \ldots, g_k \in X^n_{d_1,\ldots,d_r}$ be general polynomial vectors and $f \in \langle g_1, \ldots, g_k \rangle$ general. Let $T = \langle T_{g_1} X^n_{d_1,..,d_r}, .., T_{g_k} X^n_{d_1,..,d_r} \rangle$. If

- $\dim T = k(n+r)$
- $\mathcal{C}_k = \{g \in X^n_{d_1,\ldots,d_r} \mid T_g X^n_{d_1,..,d_r} \subset T\}$ is 0-dim at each g_i

then f is Waring identifiable over $\mathbb{C}$ with rank k.

Hessian criterion can be implemented in Macaulay2 ([M2]).

A (simultaneous) Waring decomposition has *sub-generic rank k* if:

$$k < \frac{\sum_{j=1}^{r} \binom{n+d_j}{d_j}}{n+r}. \quad (5)$$

Hessian criterion based on a generalization of Lemma 5.1, [COV1].

Lemma (Sufficient condition for identifiability)

Let $X^n_{d_1,\dots,d_r} = \mathbb{P}(\bigoplus_{j=1}^r \mathcal{O}_{\mathbb{P}^n_{\mathbb{C}}}(d_j))$ be not k-defective, with k as in (5). Let $g_1, \dots, g_k \in X^n_{d_1,\dots,d_r}$ be general polynomial vectors and $f \in \langle g_1, \dots, g_k \rangle$ general. Let $T = \langle T_{g_1} X^n_{d_1,..,d_r}, .., T_{g_k} X^n_{d_1,..,d_r} \rangle$. If

- $\dim T = k(n+r)$
- $\mathcal{C}_k = \{g \in X^n_{d_1,\dots,d_r} \mid T_g X^n_{d_1,..,d_r} \subset T\}$ is 0-dim at each g_i

then f is Waring identifiable over $\mathbb{C}$ with rank k.

Hessian criterion can be implemented in Macaulay2 ([M2]).

A (simultaneous) Waring decomposition has *sub-generic rank k* if:

$$k < \frac{\sum_{j=1}^{r} \binom{n+d_j}{d_j}}{n+r}. \qquad (5)$$

Hessian criterion based on a generalization of Lemma 5.1, [COV1].

Lemma (Sufficient condition for identifiability)

Let $X^n_{d_1,\dots,d_r} = \mathbb{P}(\bigoplus_{j=1}^{r} \mathcal{O}_{\mathbb{P}^n_{\mathbb{C}}}(d_j))$ be not k-defective, with k as in (5). Let $g_1, \dots, g_k \in X^n_{d_1,\dots,d_r}$ be general polynomial vectors and $f \in \langle g_1, \dots, g_k \rangle$ general. Let $T = \langle T_{g_1} X^n_{d_1,..,d_r}, .., T_{g_k} X^n_{d_1,..,d_r} \rangle$. If

- $\dim T = k(n+r)$
- $\mathcal{C}_k = \{g \in X^n_{d_1,\dots,d_r} \mid T_g X^n_{d_1,..,d_r} \subset T\}$ is 0-dim at each g_i

then f is Waring identifiable over $\mathbb{C}$ with rank k.

Hessian criterion can be implemented in Macaulay2 ([M2]).

INPUT $n, r, d_1, \ldots, d_r, k$

PROCEDURE

1) construct a parametrization of $X^n_{d_1,\ldots,d_r} \subset \mathbb{P}^{N-1}$, where $N = \sum_{j=1}^{r} \binom{n+d_j}{d_j}$

2) choose k random points $g_1, \ldots, g_k \in X^n_{d_1,\ldots,d_r}$ and construct the $k(n+r) \times N$ matrix j_1 associated to T
   2a) if $\operatorname{rk} j_1 = k(n+r)$ then go to step 3)
   2b) otherwise $k := k-1$ and come back to step 2)

3) determine the Cartesian equations of T by computing $\ker j1$

4) determine the Cartesian equations of $\mathcal{C}_k$ by partial derivation of equations obtained in 3)

INPUT $n, r, d_1, \ldots, d_r, k$

PROCEDURE

1) construct a parametrization of $X^n_{d_1,\ldots,d_r} \subset \mathbb{P}^{N-1}$, where $N = \sum_{j=1}^r \binom{n+d_j}{d_j}$

2) choose k random points $g_1, \ldots, g_k \in X^n_{d_1,\ldots,d_r}$ and construct the $k(n+r) \times N$ matrix j_1 associated to T

2a) if $\mathrm{rk}\, j_1 = k(n+r)$ then go to step 3)

2b) otherwise $k := k-1$ and come back to step 2)

3) determine the Cartesian equations of T by computing $\ker j1$

4) determine the Cartesian equations of $\mathcal{C}_k$ by partial derivation of equations obtained in 3)

INPUT $n, r, d_1, \ldots, d_r, k$

PROCEDURE

1) construct a parametrization of $X^n_{d_1,\ldots,d_r} \subset \mathbb{P}^{N-1}$, where $N = \sum_{j=1}^{r} \binom{n+d_j}{d_j}$

2) choose k random points $g_1, \ldots, g_k \in X^n_{d_1,\ldots,d_r}$ and construct the $k(n+r) \times N$ matrix j_1 associated to T

   2a) if $\operatorname{rk} j_1 = k(n+r)$ then go to step 3)

   2b) otherwise $k := k-1$ and come back to step 2)

3) determine the Cartesian equations of T by computing $\ker j1$

4) determine the Cartesian equations of $\mathcal{C}_k$ by partial derivation of equations obtained in 3)

INPUT $n, r, d_1, \ldots, d_r, k$

PROCEDURE

1) construct a parametrization of $X^n_{d_1,\ldots,d_r} \subset \mathbb{P}^{N-1}$, where $N = \sum_{j=1}^r \binom{n+d_j}{d_j}$

2) choose k random points $g_1, \ldots, g_k \in X^n_{d_1,\ldots,d_r}$ and construct the $k(n+r) \times N$ matrix j_1 associated to T
   2a) if $\mathrm{rk}\, j_1 = k(n+r)$ then go to step 3)
   2b) otherwise $k := k-1$ and come back to step 2)

3) determine the Cartesian equations of T by computing $\ker j1$

4) determine the Cartesian equations of $\mathcal{C}_k$ by partial derivation of equations obtained in 3)

INPUT $n, r, d_1, \ldots, d_r, k$

PROCEDURE

1) construct a parametrization of $X^n_{d_1,\ldots,d_r} \subset \mathbb{P}^{N-1}$, where $N = \sum_{j=1}^r \binom{n+d_j}{d_j}$

2) choose k random points $g_1, \ldots, g_k \in X^n_{d_1,\ldots,d_r}$ and construct the $k(n+r) \times N$ matrix j_1 associated to T
   - 2a) if $\operatorname{rk} j_1 = k(n+r)$ then go to step 3)
   - 2b) otherwise $k := k-1$ and come back to step 2)

3) determine the Cartesian equations of T by computing $\ker j1$

4) determine the Cartesian equations of $\mathcal{C}_k$ by partial derivation of equations obtained in 3)

INPUT $n, r, d_1, \ldots, d_r, k$

PROCEDURE

1) construct a parametrization of $X^n_{d_1,\ldots,d_r} \subset \mathbb{P}^{N-1}$, where $N = \sum_{j=1}^{r} \binom{n+d_j}{d_j}$

2) choose k random points $g_1, \ldots, g_k \in X^n_{d_1,\ldots,d_r}$ and construct the $k(n+r) \times N$ matrix j_1 associated to T
   2a) if $\operatorname{rk} j_1 = k(n+r)$ then go to step 3)
   2b) otherwise $k := k-1$ and come back to step 2)

3) determine the Cartesian equations of T by computing $\ker j1$

4) determine the Cartesian equations of $\mathcal{C}_k$ by partial derivation of equations obtained in 3)

5) determine the dimension of $\mathcal{C}_k$ at g_i by computing codim $T_{g_i}\mathcal{C}_k$, i.e. the rank of the Jacobian H of equations in 4) evaluated at $g_i \rightsquigarrow H =$ *hessian* matrix of equations in 3)

5a) if $\operatorname{rk} H = n + r - 1$ then $X^n_{d_1,\ldots,d_r}$ is k-identifiable over $\mathbb{C}$

5b) otherwise $k := k - 1$ and come back to step 2)

OUTPUT

- 5a) $\rightsquigarrow k$
- 2b) $\rightsquigarrow (k, k(n + r) - \operatorname{rk} j_1)$
- 5b) $\rightsquigarrow (k, \operatorname{rk} H, \dim \mathcal{C}_k)$

5) determine the dimension of $\mathcal{C}_k$ at g_i by computing codim $T_{g_i}\mathcal{C}_k$, i.e. the rank of the Jacobian H of equations in 4) evaluated at $g_i \rightsquigarrow H =$ *hessian* matrix of equations in 3)

5a) if $\operatorname{rk} H = n + r - 1$ then $X^n_{d_1,\ldots,d_r}$ is k-identifiable over $\mathbb{C}$

5b) otherwise $k := k - 1$ and come back to step 2)

OUTPUT

- 5a) $\rightsquigarrow k$
- 2b) $\rightsquigarrow (k, k(n+r) - \operatorname{rk} j_1)$
- 5b) $\rightsquigarrow (k, \operatorname{rk} H, \dim \mathcal{C}_k)$

5) determine the dimension of $\mathcal{C}_k$ at g_i by computing codim $T_{g_i}\mathcal{C}_k$, i.e. the rank of the Jacobian H of equations in 4) evaluated at $g_i \rightsquigarrow H =$ *hessian* matrix of equations in 3)

5a) if $\operatorname{rk} H = n + r - 1$ then $X^n_{d_1,\dots,d_r}$ is k-identifiable over $\mathbb{C}$

5b) otherwise $k := k - 1$ and come back to step 2)

<u>OUTPUT</u>

- 5a) $\rightsquigarrow k$
- 2b) $\rightsquigarrow (k, k(n+r) - \operatorname{rk} j_1)$
- 5b) $\rightsquigarrow (k, \operatorname{rk} H, \dim \mathcal{C}_k)$

J. Alexander, A. Hirschowitz
Polynomial interpolation in several variables, J. Algebraic Geom., 4 (2) (1995), 201-222

A. Anandkumar, R. Ge, D. Hsu, S.M. Kakade, M. Telgarsky
Tensor decompositions for learning latent variable models, J. Mach. Learn. Res., 15 (2014), 2773-2832

E. Angelini
Waring decompositions and identifiability via Bertini and Macaulay2 softwares, (2017), in preparation

E. Angelini, C. Bocci, L. Chiantini
Catalecticant intersections and non-identifiable cases, (2017), in preparation

E. Angelini, C. Bocci, L. Chiantini
Real identifiability vs complex identifiability, (2016), arXiv:1608.07197 [math.AG]

E. Angelini, F. Galuppi, M. Mella, G. Ottaviani
On the number of Waring decompositions for a generic polynomial vector, (2016), to appear on Journal of Pure and Applied Algebra, arXiv:1601.01869 [math.AG]

C.J. Appellof, E.R. Davidson
Strategies for analyzing data from video fluorometric monitoring of liquid chromatographic effluents, Anal. Chem., 53 (13) (1981), 2053-2056

D. J. Bates, J. D. Hauenstein, A. J. Sommese, C. W. Wampler,
Bertini: Software for numerical algebraic geometry, available at `bertini.nd.edu`

D. J. Bates, J. D. Hauenstein, A. J. Sommese, C. W. Wampler,
Numerically Solving Polynomial Systems with Bertini, SIAM, Philadelphia (2013)

L. Chiantini, G. Ottaviani, N. Vannieuwenhoven
Effective criteria for specific identifiability of tensors and forms, (2016), to appear on SIAM J. Matrix Anal. Applic. arXiv:1609.00123

L. Chiantini, G. Ottaviani, N. Vannieuwenhoven
On generic identifiability of symmetric tensors of sub-generic rank, Trans. Amer. Math. Soc. 369 (2017), 4021-4042

L. Chiantini, G. Ottaviani, N. Vannieuwenhoven
An algorithm for generic and low-rank specific identifiability of complex tensors, SIAM J Matrix Anal Appl, 35 (2014), 1265-1287

P. Comon, L.H. Lim, Y. Qi
Semialgebraic geometry of nonnegative tensor rank, (2016), arXiv:1601.05351 [math.AG]

C. Ciliberto, F. Russo, *Varieties with minimal secant degree and linear systems of maximal dimension on surfaces*, Advances in Mathematics 200 (2006) 1–50

I. Domanov, L. De Lathauwer
Generic uniqueness conditions for the canonical polyadic decomposition and INDSCAL, SIAM J Matrix Anal Appl, 36 (4) (2015), 1567-1589

- A. Dixon, T. Stuart, *On the reduction of the ternary quintic and septimic to their canonical forms*, Proc. London Math. Soc. (2) 4 (1906) 160–168
- F. Galuppi, M. Mella, *Identifiability of homogeneous polynomials and Cremona Transformations*, (2016) arXiv:1606.06895 [math.AG]
- D. Grayson, M. Stillman, *Macaulay2, a software system for research in algebraic geometry*, available at `www.math.uiuc.edu/Macaulay2`
- D. Hilbert, *Letter adresseé à M. Hermite*, Gesam. Abh. vol II, 148–153
- M. Michalek, H. Moon, B. Sturmfels, E. Ventura, *Real rank geometry of ternary forms*, (2016) arXiv:1606.06895 [math.AG]

- K. Ranestad and F. Schreyer. *Varieties of sums of powers*, J. Reine Angew. Math. 525 (2000), 147–181
- R.A. Roberts, *Note on the plane cubic and a conic*, Proc. London Math. Soc. (1) 21, (1889), 62–69
- J.J. Sylvester, *Collected Works*, Cambridge University Press, 1904
- K. Weierstrass, *Zur theorie der bilinearen and quadratischen Formen*, Monatsh. Akad. Wisa. Berlin, (1867), pp. 310–338